Teacher Note

Reasoning and Proof in Mathematics

As students find strategies to perform calculations, they frequently make claims about numerical relationships. Part of the work of fourth grade involves helping students strengthen their ability to verbalize those claims and to consider questions such as these: Does this claim hold for *all* numbers? How can we know? Finding ways to answer these questions provides the basis for making sense of formal proof when it is introduced years from now. Consider the following vignette, in which a fourth-grade class is discussing methods for solving multiplication problems.

Andrew: When I did 12 × 25, I cut the 12 in half and doubled the 25 to make it 6 × 50. I can do 6 × 50 in my head. It's 300.

Teacher: Did anyone else use a strategy like Andrew's on any of the problems?

Lucy: I did. I was working on 14 × 15. I did the same kind of thing. I changed it to 7 × 30 and that's 210.

Teacher: Let's look at this. Are you saying that 12 × 25 = 6 × 50? And that 14 × 15 = 7 × 30?

Sabrina: The product stays the same. You cut one number in half and you double the other, so the answer is the same.

Teacher: Are you saying that this *always* works—that when you multiply two numbers, you can cut one number in half and double the other number, and the product will stay the same? Does the product stay the same no matter what the numbers?

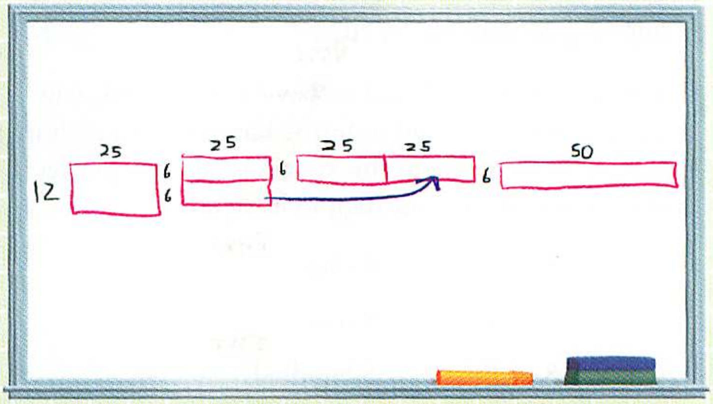
Sabrina: I think so.

Teacher: Can we find a way to use diagrams or cubes to show what is happening and why the product stays the same? You may begin with the examples we have seen, but you also must show how your argument will work for *all* numbers.

In this class, Sabrina made an assertion—mathematicians call such an assertion a *conjecture*—that the product of two numbers remains the same if you divide one of the numbers by 2 and double the other number. The teacher has challenged the class to find a way to show that this conjecture is true—not just for the examples they have noted, but for *all* pairs of numbers. If they can find a proof, they have what mathematicians call a *theorem* or *proposition*.

Let us return to the Grade 4 classroom to see how the students responded to their teacher's challenge to justify their conjecture.

Noemi: I made a diagram to show it.



Noemi: For Andrew's example, the first rectangle is 12 by 25. Then he cut the 12 in half and moved the bottom rectangle to make a rectangle that is 6 by 50. The first rectangle has the same area as the last one because they're both made up of the same two smaller rectangles. So like Andrew says, 12 × 25 = 6 × 50.

Lucy: Noemi's picture works for my example too. You can think of the first rectangle as 14 × 15 and the last rectangle as 7 × 30.

Noemi: It doesn't matter what the numbers are. If you cut one in half, you would always have a rectangle on the bottom to move, and when you move that rectangle, you make a rectangle that is twice as long. It works for all numbers.

Noemi has presented a model of multiplication to show how she knows that 12 × 25 = 6 × 50. Lucy sees that the same representation can be used to show $14 \times 15 = 7 \times 30$. Noemi then points out that, in fact, it doesn't matter what the numbers are. The model applies to *any* two numbers that are multiplied (provided that they are positive).

Note: The fact that Sabrina's conjecture has been shown to be true—if you double one factor and halve the other, you maintain the same product—does not necessarily mean that this doubling/halving strategy makes all multiplication easier. If both numbers are odd (e.g., 17×13), halving and doubling will result in one number that includes a fractional part (e.g., $34 \times 6\frac{1}{2}$). This is not a strategy one would choose to solve all multiplication problems. However, if both numbers are even, halving and doubling to create an equivalent problem can often lead to a simpler computation. Just as important, exploring *why* doubling and halving (and tripling and thirding, etc.) work provides an opportunity to learn more about the properties of multiplication and about developing mathematical justification.

Students in Grades K–5 can work productively on developing justifications for mathematical ideas, as Noemi does here. But what is necessary to justify an idea in mathematics? First we'll examine what "proof" is in the field of mathematics and then return to what kind of justification students can do in fourth grade.

What Is "Proof" in Mathematics?

Throughout life, when people make a claim or assertion, they are often required to justify the claim, to persuade others. A prosecutor who claims that a person is quilty must make an argument, based on evidence, to convince the jury of this claim. A scientist who asserts that the earth's atmosphere is becoming warmer must marshal evidence, usually in the form of data and accepted theories and models, to justify the claim. Every field, including the law, science, and mathematics, has its own accepted standards and rules for how a claim must be justified to persuade others.

When students in Grades K–5 are asked to give reasons why their mathematical claims are true, they often say things like this: "It worked for all the numbers we could think of." "I kept on trying and it kept on working." "We asked the sixth graders and they said it was true." "We asked our parents." These are appeals to particular instances and to authority. In any field, there are appropriate times to turn to authority (a teacher or a book, for example) for help with new knowledge or with an idea that we don't yet have enough experience to think through for ourselves. Similarly, particular examples can be very helpful in understanding some phenomenon. However, neither an authoritative statement nor a set of examples is sufficient to prove a mathematical assertion about an infinite class (say, all whole numbers).

In mathematics, a *theorem* must start with a mathematical assertion, which has explicit hypotheses ("givens") and an explicit conclusion. The proof of the theorem must show how the conclusion follows logically from the hypotheses. For instance, the fourth graders asserted that the product of two numbers remains the same if you divide one of the numbers by 2 and double the other number. In later years, their theorem might be stated as: If m and n are numbers, $m \times n = (\frac{m}{2}) \times (n \times 2)$. The proof of this claim consists of a series of steps in which one begins with the hypothesis—m and n are numbers—and follows a chain of logical deductions ending with the conclusion—$m \times n = (\frac{m}{2}) \times (n \times 2)$. Each deduction must be justified by an accepted definition, fact, or principle, such as the commutative or associative property of multiplication.

For example, to show that $m \times n = (m \times \frac{1}{2}) \times (n \times 2)$, we can develop this set of steps:

$$\begin{aligned} m \times n &= [m \times (\tfrac{1}{2} \times 2] \times n \\ &= [(m \times \tfrac{1}{2}) \times 2] \times n \\ &= (m \times \tfrac{1}{2}) \times (2 \times n) \end{aligned}$$

In this series of steps, the associative property of multiplication is applied twice. The associative property can be written with symbolic notation as $(a \times b) \times c = a \times (b \times c)$; regrouping the factors does not affect the product. For example, in the series of steps above, $m \times (\frac{1}{2} \times 2)$ can be regrouped as $(m \times \frac{1}{2}) \times 2$. It may help to look at how this works with one of the examples from the classroom dialogue:

$$\begin{aligned} 12 \times 25 &= [12 \times (\tfrac{1}{2} \times 2)] \times 25 \\ &= [(12 \times \tfrac{1}{2}) \times 2] \times 25 \\ &= (12 \times \tfrac{1}{2}) \times (2 \times 25) = 6 \times 50 \end{aligned}$$

The model for such a notion of proof was first established by Euclid, who codified what was known of ancient Greek geometry in his *Elements,* written about 300 B.C. In his book, Euclid begins with the basic terms and postulates of geometry and, through hundreds of propositions and proofs, moves to beautiful and surprising theorems about geometric figures. What is remarkable is that, in each mathematical realm, you can get so far with such simple building blocks.

What Does Proof Look Like in Fourth Grade?

One does not expect the rigor or sophistication of a formal proof, or the use of algebraic symbolism, from young children. Even for a mathematician, precise validation is often developed *after* new mathematical ideas have been explored and are solidly understood. When mathematical ideas are evolving and there is a need to communicate the sense of *why* a claim is true, then informal means of justification are appropriate. Such a justification can include the use of visual displays, concrete materials, or words. The test of the effectiveness of such a justification is this: Does it rely on logical thinking about the mathematical relationships rather than on the fact that one or a few specific examples work?

This informal approach to mathematical justification is particularly appropriate in Grade K–5 classrooms, where mathematical ideas are generally "under construction" and where sense-making and diverse modes of reasoning are valued. Noemi's argument offers justification for the claim that if you halve one factor and double the other, the product remains the same. The product of the numbers m and n is represented by the area of a rectangle with dimensions m and n. Noemi then cuts the vertical dimension in half, making two rectangles, each having dimensions $(\frac{m}{2})$ and n. One of these rectangles is moved and then connected with the other to create a rectangle with dimensions $(\frac{m}{2})$ by $2 \times n$. The area of this new rectangle must be the same as the original, therefore $(\frac{m}{2}) \times (2 \times n) = m \times n$. Noemi's argument establishes the validity of the claim not only for particular numbers, but for any numbers, and easily conveys why it is true.

An important part of Noemi's justification is her statement that it does not matter what the numbers are. She understands that the process she describes with her model will guarantee that the original rectangle will have the same area as the final rectangle whose length is double that of the original and whose width is half that of the original. It is important to note that when students make such claims of generality—*this is true for all numbers*—the phrase *all numbers* refers to the numbers they are using. In this vignette, Noemi's reasoning about multiplication takes place in the context of whole numbers. We might see that Noemi's argument applies equally well to positive values that include rational numbers, but Noemi and her classmates will need to revisit this argument when the domain of numbers they are working with expands beyond whole numbers.

To support the kind of reasoning illustrated in the vignette, encourage students to use cubes, number lines, and other representations to explain their thinking. The use of representations offers a reference for the student who is explaining his or her reasoning, and it also allows more classmates to follow that reasoning. If it seems that students may be thinking only in terms of specific numbers, you might ask such questions as these: Will that work for other numbers? How do you know? Will the explanation be the same?