Teacher Note

Images of Multiplication

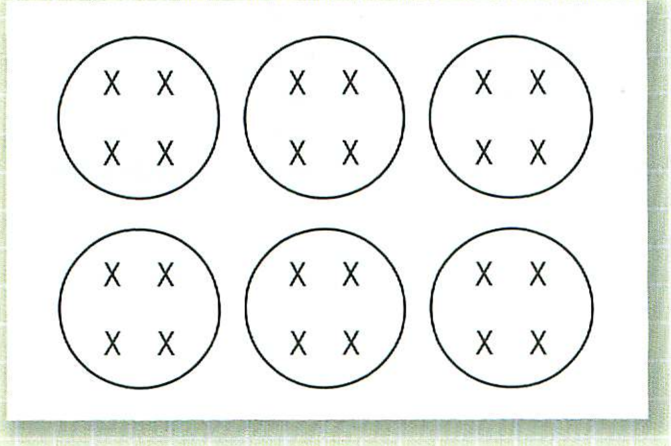
It is important that students develop strong visual images of multiplication as they develop strategies for solving multiplication problems. If students can visualize clearly how the numbers they are multiplying are related, they can develop flexible, efficient, and accurate strategies for solving multiplication problems.

Students encountered many ways to represent multiplication in Grade 3 that they will continue to use in Grade 4—pictures of groups of things in a story context, skip counting on a 100 chart, and arrays. As students work with larger numbers in Grade 4, it becomes cumbersome to draw pictures, skip count on a large number chart, or use arrays with all the individual units shown. Students need to learn to visualize these representations mentally to help them break up the numbers and keep track of which parts of the problem have been solved and which remain to be solved.

As you work with students, suggest these ways of visualizing multiplication, especially when a student cannot figure out where to start or when a student has solved part of a problem and is unsure how to continue.

Images of Equal Groups in a Story Context

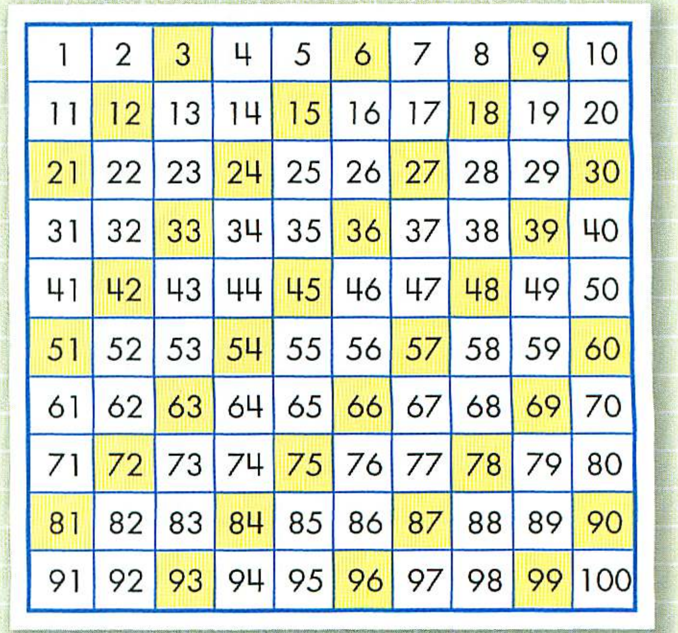
Most students can represent a multiplication expression such as 6 × 4 by creating a picture similar to this one:



Ask students to generate simple stories that help them visualize a multiplication expression such as 6 × 4 as equal groups (e.g., 6 bags with 4 marbles in each bag). Help students select simple contexts that are familiar to them. Then you can ask students to imagine that context as a way of thinking through the problem. In Grades 3 and 4, students are moving away from thinking of multiplication as repeated addition. Instead of adding up 4s, students can be encouraged to use the image to start with a larger chunk of the problem. For example, you might ask, "Can you visualize how many marbles would be in 2 bags? In 3 bags? . . . Now how many more bags of 4 are there?" A story context involving equal groups can help students use what they know to determine the product: "I know that there are 12 marbles in 3 of the bags—oh, so there's 3 more bags, so I double that to get 24."

Representing Multiplication as Skip Counting

In Grade 3, students spent time creating and understanding skip counting charts. On 100 charts, students marked off multiples of the numbers 2 through 12.

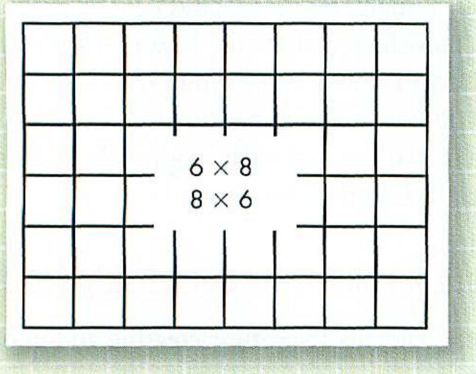


1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

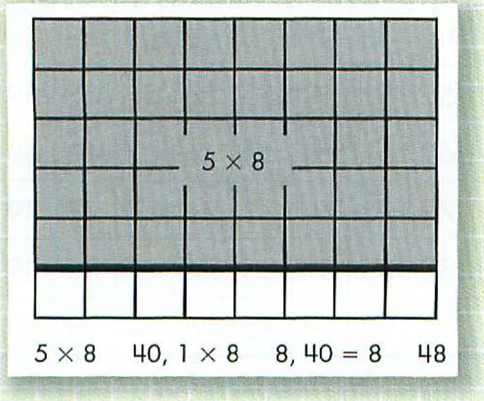
These charts provided an opportunity for students to notice patterns in each number's multiples and to consider the relationship between multiples of various numbers. Ask students questions that help them visualize the counting number sequence and think through how to calculate the next multiple as they are skip counting.

Representing Multiplication with Arrays

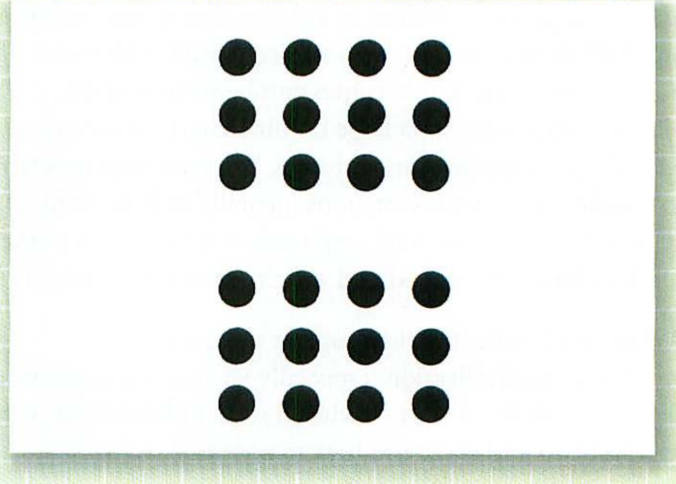
In this unit, students work with Array Cards and drawings in which all the individual units of the array are visible. These arrays are a representation of the groups and amounts in a group in any multiplication problem.



This 6 × 8 array can be seen as 8 groups of 6 items or as 6 groups of 8 items. In either case, students can visualize the problem as a whole and then visualize the smaller parts that may help them find the product. For example, if juice boxes come in sets of 6, a student might think of 8 × 6 as 8 sets of juice boxes. The student could visualize these in an array and use that image to break the problem into parts that are easier to solve, as follows:



The story problems in Investigations 1 and 2 and the *Quick Images* activity that is introduced in this unit (page 59) also provide experience with arrays that represent multiplication situations. For example, *Student Activity Book* page 1 shows cans in a 6 × 8 array. The *Quick Images* in this unit show combinations of arrays that provide the opportunity to describe a product in different ways. For example, *Quick Image* 7 might be described as 6 × 4 (6 rows of 4 dots), 8 × 3 (8 groups of vertical rows of 3 dots each), 2 × 12 (2 rectangles, each with 12 dots), or even 2 × 3 × 4 (2 rectangles, each with 3 rows of 4 dots).



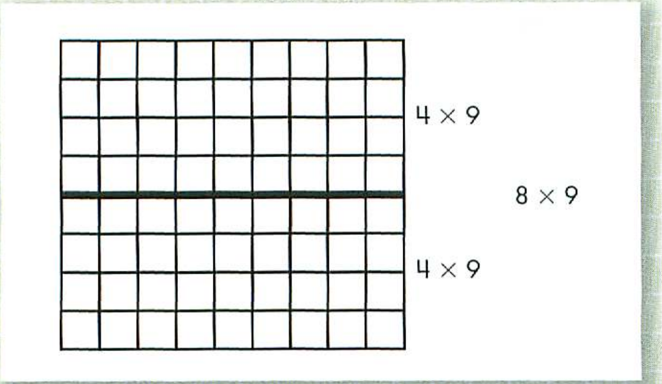
Visualizing how to break multiplication problems into parts becomes even more important as students solve multidigit problems in *Multiple Towers and Division Stories* and *How Many Packages? How Many Groups?* See **Teacher Note:** Representing Multiplication with Arrays (page 117), for more information about how arrays are used in this unit and how the use of arrays can be extended to represent more difficult multiplication and division problems. In the next unit on multiplication and division, *Multiple Towers and Division Stories,* the **Teacher Note:** Visualizing Arrays provides information about how and why students make a transition from using arrays marked with individual units to visualizing unmarked arrays.

Representing Multiplication with Arrays

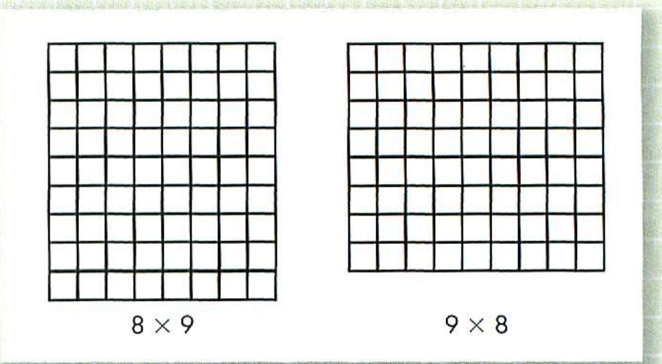
Representing mathematical relationships is a key element of developing mathematical understanding. For multiplication, the rectangular array is an important tool. It meets all the criteria for a powerful mathematical representation: it highlights important relationships, provides a tool for solving problems, and can be extended as students apply ideas about multiplication in new areas.

Why Arrays for Multiplication?

As students come to understand the operation of multiplication in Grades 3 and 4, they gradually move away from thinking of multiplication as only repeated addition. They learn that multiplication has particular properties that distinguish it from addition. Although a number line or 100 chart can be used to show how multiplication can be viewed as adding equal groups, neither of these tools provides easy access to other important properties of multiplication. The rectangular array provides a window into properties that are central to students' work in learning the multiplication combinations and in solving multidigit multiplication and division problems.

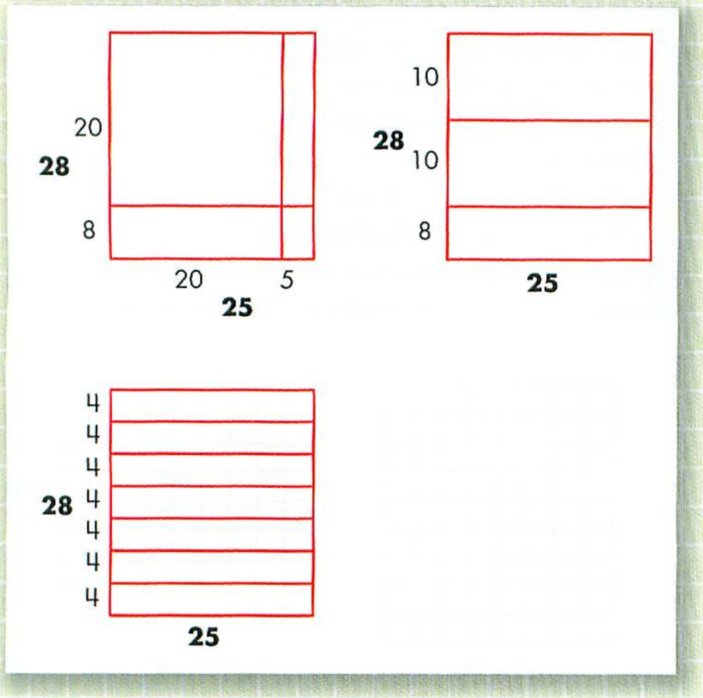


Here is an example—working on 8 × 9, one of the more difficult multiplication combinations for most students—of how an array can illustrate that 8 × 9 = (4 × 9) + (4 × 9). In splitting a multiplication problem such as 8 × 9 into sub problems [(4 × 9) + (4 × 9)], you are using the distributive property. The number that you break up is distributed into parts that must all be multiplied by the other number. This property of multiplication is at the core of almost all common strategies used to solve multiplication problems.

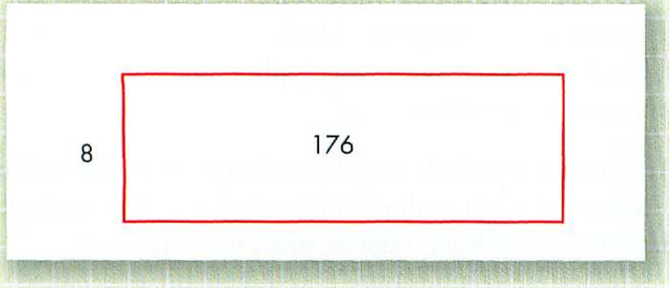


The rectangular array also makes it clearer why the product of 9 × 8 is the same as the product of 8 × 9. The array can be rotated to show that 9 rows with 8 in each row have the same number of squares as 8 rows with 9 in each row. The column on one becomes the row on the other, illustrating the commutative property—the fact that you can change the order of the factors in a multiplication equation without changing the product.

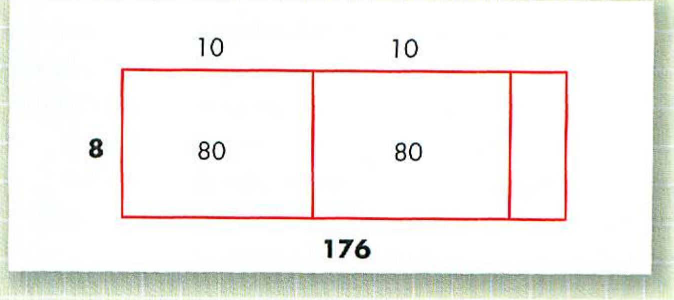
Arrays are particularly useful for solving or visualizing how to solve multidigit multiplication problems. After students have worked with rectangular arrays for single-digit multiplication combinations and thoroughly understand how an array represents the factors and product, they can use arrays in their work to solve harder problems later in Grade 4. For example, the array for 28 × 25 can be broken up in many ways, as follows:



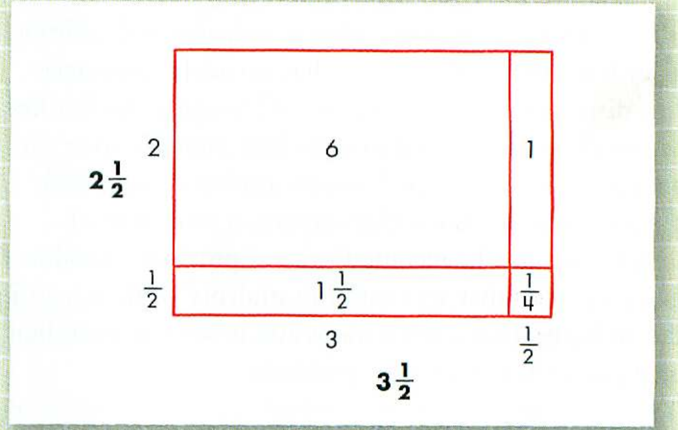
Arrays also support students' learning about the relationship between multiplication and division. In a division problem such as 176 ÷ 8, the dividend (176) is represented by the number of squares in the array, and the divisor (8) is one dimension of the array.

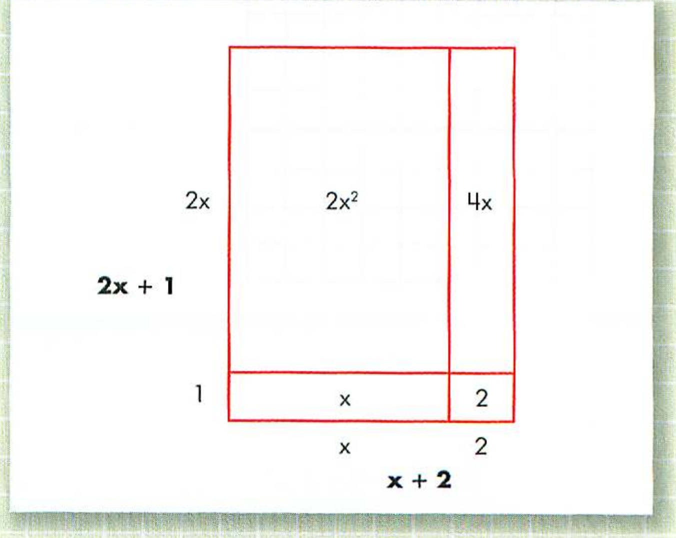


Students can think of "slicing off" pieces of the rectangle as they gradually figure out the other factor:

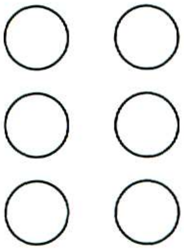


Finally, the use of the rectangular array can be extended in later grades as students work with multiplication of fractions and, later, of algebraic expressions.





For multiplication notation to describe arrays, the *Investigations* curriculum uses the convention of designating the number of rows first and the number in each row second; for example, 3 × 2 indicates 3 rows with 2 in each row.

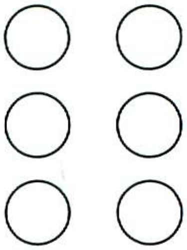


3 × 2

"There are 3 rows with 2 circles in each row."

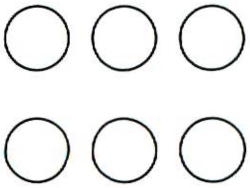
This convention is consistent with using 3 × 2 to indicate 3 groups of 2 in other multiplication situations (e.g., 3 pots with 2 flowers in each pot). However, at this age level, it is not necessary for students to follow this system rigidly; trying to remember which number stands for rows and which for the number in a row can be unnecessarily distracting for students.

When students suggest a multiplication expression for an array, what is important is that they understand what the numbers mean; for example, a student might show how 3 × 2 represents 3 rows of cans with 2 in each row or 3 cans in each of 2 rows.



3 × 2

"There are 3 rows with 2 circles in each row."



3 × 2

"See, there are 3 here and 3 here, so you double the 3."

Note that in other cultures, conventions about interpreting multiplication expressions differ. In some countries, the convention for interpreting 3 × 2 is not "3 groups of 2" but "3 taken 2 times."

Learning and Assessing Multiplication Combinations

In Investigation 2 of this unit, students review the multiplication combinations they worked on in Grade 3 (combinations with products to 50) and then work on the rest of the multiplication combinations to 12 × 12 so that they become fluent with all of them. Students are expected to know the multiplication combinations fluently by the middle of Grade 4 (see the list of benchmarks by grade level at the end of this Teacher Note).

Although there is some time built into Investigation 2 to work on the multiplication combinations, many students will need additional practice during this unit and after this unit is completed. You can use Multiplication Combinations (M51) as well as the Multiplication Cards (M35–M40) for this practice. With your guidance about which multiplication combinations to work on, students should continue learning these combinations at home or outside of math time. You may have other favorite practice methods or activities that you want to suggest for particular students. Also, enlist the help of parents or other family members. This Teacher Note provides recommendations for supporting students in their ongoing practice.

Why Do We Call Them *Combinations*?

The pairs of factors from 1 × 1 through 12 × 12 are traditionally referred to as "multiplication facts"—those multiplication combinations with which students are expected to be fluent. The *Investigations* curriculum follows the National Council of Teachers of Mathematics (NCTM) convention of calling these expressions *combinations* rather than *facts*. *Investigations* does this for two reasons. First, naming *only* particular addition and multiplication combinations as *facts* seems to give them elevated status, more important than other critical parts of mathematics. In addition, the word *fact* implies that something cannot be learned through reasoning. For example, it is a fact that the first president of the United States was George Washington, and it is a fact that Rosa Parks was born in Alabama in 1913. If these facts are important for us to know, we can remember them or use reference materials to look them up. However, the product for the multiplication combination 6 × 7 can be determined in many ways; it is logically connected to our system of numbers and operations. If we forget the product, but understand what multiplication is and know some related multiplication combinations, we can find the product through reasoning. For example, if we know that $5 \times 7 = 35$, we can add one more 7 to determine that the product of 6 × 7 is 42. If we know that $3 \times 7 = 21$, we can reason that the product of 6 × 7 would be twice that, $2 \times (3 \times 7) = 42$.

The term *facts* does convey a meaning that is generally understood by some students and family members, so you will need to decide whether to use the term *facts* along with *combinations* in certain settings in order to make your meaning clear.

Fluency with Multiplication Combinations

Like NCTM, this curriculum supports the importance of students learning the basic combinations through a focus on reasoning about number relationships: "Fluency with whole-number computation depends, in large part, on fluency with basic number combinations—the single digit addition and multiplication pairs and their counterparts for subtraction and division. Fluency with basic number combinations develops from well-understood meanings for the four operations and from a focus on thinking strategies. . . ." (*Principles and Standards for School Mathematics*, pages 152–153)

Fluency means that combinations are quickly accessible mentally, either because they are immediately known or because the calculation that is used is so effortless as to be essentially automatic (in the way that some adults quickly derive one combination from another).

Helping Students Learn the Multiplication Combinations

A. Students Who Know Their Combinations to 50

Students who know their combinations to 50, as well as the combinations that involve multiplying by 10 up to 100 (6 × 10, 7 × 10, 8 × 10, 9 × 10, 10 × 10), can work on learning the most difficult combinations. Here is one way of sequencing this work.

1. Learning the remaining combinations with products to 100. There are 6 difficult facts to learn (other than the ×11 and ×12 combinations, which are, in fact, not as difficult as these, and are discussed below). These six difficult combinations are: 6 × 9 (and 9 × 6), 7 × 8 (and 8 × 7), 7 × 9 (and 9 × 7), 8 × 8, 8 × 9 (and 9 × 8), and 9 × 9. Note that knowing that multiplication is commutative is crucial for learning all the multiplication combinations. The work with Array Cards supports this understanding, see **Teacher Note:** Representing Multiplication with Arrays, page 117.

Students can work on one or two of these most difficult multiplication combinations each week. Make sure that they use combinations they do know to help them learn ones they don't know—for example, $8 \times 7 = 2 \times (4 \times 7)$, or $9 \times 7 = (10 \times 7) - 7$. They can write these related multiplication combinations as "start with" hints on the Multiplication Cards. If most of your class needs to work on the same few hard combinations, you might want to assign the whole class to focus on two of these each week.

2. Learning the ×11 and ×12 combinations. We consider these combinations to be in a different category. Historically, these combinations were included in the list of "multiplication facts." However, when we are dealing with 2-digit numbers in multiplication, an efficient way to solve them is through applying the distributive property, breaking the numbers apart by place as you would with any other 2-digit numbers. We include them here because some local or state frameworks still require knowing multiplication combinations through 12 × 12. In addition, 12 is a number that occurs often in our culture, and it is useful to know the ×12 combinations fluently. Most students learn the ×11 combinations easily because of the pattern (11, 22, 33, 44, 55, . . .) created by multiplying successive whole numbers by 11. They should also think through why this pattern occurs: $3 \times 11 = (3 \times 10) + (3 \times 1) = 30 + 3 = 33$. They should think through why $11 \times 10 = 110$ and $11 \times 11 = 121$ by breaking up the numbers. Students can learn the ×12 combinations by breaking the 12 into a 10 and 2, e.g., $12 \times 6 = (10 \times 6) + (2 \times 6)$. Some students may also want to use doubling or adding on to known combinations: $12 \times 6 = 2 \times (6 \times 6)$, or $12 \times 6 = (11 \times 6) + 6$.

B. Students Who Need Review and Practice of Combinations to 50

Students who have difficulty learning the multiplication combinations often view this task as overwhelming—an endless mass of combinations with no order and reason. Bringing order and reason to students' learning of these combinations in a way that lets them have control over their progress is essential. Traditionally, students learned one "table" at a time (e.g., first the ×2 combinations, then the ×3 combinations, the ×4 combinations, and so on). However, the multiplication combinations can be grouped in other ways to support learning related combinations.

First, make sure that students know all multiplication combinations that involve ×0, ×1, ×2, ×5, and ×10 (up to 10 × 10) fluently. (Students worked with the ×0 combinations in Grade 3.) Note that, although most fourth graders can easily count by 2, 5, and 10, the student who is fluent does not need to skip count to determine the product of multiplication combinations involving these numbers.

When students know these combinations, turn to those that they have not yet learned. Provide a sequence of small groups of combinations that students can relate to what they already know. There are a number of ways to do this.

1. Learning the ×4 combinations. Work on the ×4 combinations that students do not yet know: 3 × 4, 4 × 4, 6 × 4, 7 × 4, 8 × 4, and 9 × 4. Help students think of these as doubling the ×2 combinations. So, $4 \times 6 = (2 \times 6) + (2 \times 6)$, or $4 \times 6 = 2 \times (2 \times 6)$. Students may verbalize this idea as "4 times 6 is 2 times 6 and another 2 times 6," or "to get 4 times 6, I double 2 × 6." Doubling is also useful within the ×4 combinations; for example, when students know that 3 × 4 = 12, then that fact can be used to solve 6 × 4: 6 × 4 = (3 × 4) + (3 × 4). Getting used to thinking about doubling with smaller numbers will also prepare students for using this approach with some of the harder combinations.

2. Learning the square numbers. Next, students learn or review the four remaining combinations that produce square numbers less than 50: 3 × 3, 5 × 5, 6 × 6, and 7 × 7. These are often easy for students to remember. If needed, use doubling or a known combination for "start with" clues during practice (e.g., 6 × 6 is double 3 × 6; 5 × 5 is 5 more than 4 × 5). Students can also build these combinations with tiles or draw them on grid paper to see how they can be represented by squares.

3. Learning the remaining combinations with products to 50. Finally, learn or review the six remaining combinations with products to 50: 3 × 6 through 3 × 9, 7 × 6, and 8 × 6. First, relate them to known combinations (e.g., double 3 × 3 or halve 6 × 6 to get 3 × 6), and then practice them.

Assessing Students' Knowledge of Multiplication Combinations

Over the next few months, do some periodic assessment to help you and your students keep track of which multiplication combinations they know fluently and which they still need to practice.

In the second multiplication and division unit of Grade 4, *Multiple Towers and Division Stories,* students will have a final check of their fluency with multiplication combinations and begin to work on their division counterparts. Work on these related division problems will continue in the last Grade 4 multiplication and division unit, *How Many Packages? How Many Groups?* In the meantime, as students work on division problems, help them relate division expressions to the multiplication combinations they know: "What multiplication combination can help you solve 24 ÷ 6?"

Fluency Benchmarks for Learning Combinations Through the Grades

Addition: fluent by end of Grade 2, with review and practice in Grade 3

Subtraction: fluent by end of Grade 3, with review and practice in Grade 4

Multiplication: fluent with multiplication combinations with products to 50 by the end of Grade 3; up to 12 × 12 by the middle of Grade 4, with continued review and practice

Division: fluent by end of Grade 5

End-of-Unit Assessment

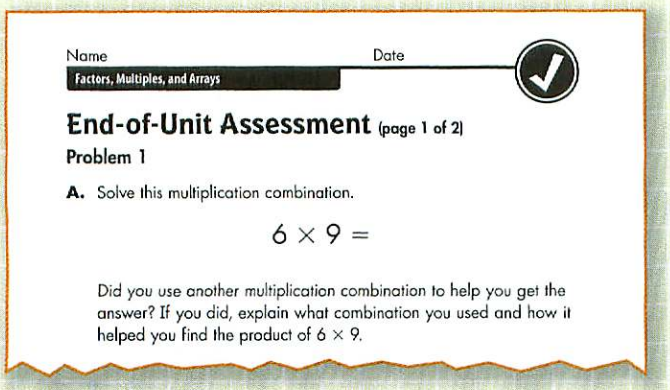
Problem 1 Part A

Benchmark addressed:

Benchmark 1: Use known multiplication combinations to find the product of any multiplication combination up to 12 × 12.

In order to meet the benchmark, students' work should show that they can:

- Demonstrate knowledge of the multiplication combination 6 × 9 or the ability to derive the product of this combination from known multiplication combinations;
- Show how they solved the problem.



Name Date

Factors, Multiples, and Arrays

End-of-Unit Assessment (page 1 of 2)

Problem 1

A. Solve this multiplication combination.

6 × 9 =

Did you use another multiplication combination to help you get the answer? If you did, explain what combination you used and how it helped you find the product of 6 × 9.

▲ Resource Masters, M55

Meeting the Benchmark

The following examples of student work provide a range of typical responses. All of these students met the benchmark—they were able to interpret the problem and solve it accurately.

Anna used known multiplication combinations to derive the product of 6 × 9 and demonstrated an understanding of 6 × 9 as the sum of the products of 6 × 6 and 6 × 3. She wrote:

"For a short cut, I did 6 × 6 to get me to 36. So then after that I just had to do 6 × 3 because 3 + 6 = 9. 6 × 3 = 18, so then I added the totals together to get 54. 36 + 18 = 54."

Derek used his understanding of the relationship between 3 groups of 9 and 6 groups of 9 to find the product of 6 × 9. He wrote:

"I know that 3 × 9 is 27. I know that 3 is half of 6, so 6 × 9 is 54. 27 + 27 = 54."

Noemi used knowledge of the ×10 combinations to solve this problem. She wrote:

"I know that 6 × 10 = 60. Then I had to take away one 6 because the problem is really 6 × 9. 60 − 6 = 54."

At this point in Grade 4, some students will know the product of 6 × 9 without having to derive it from a simpler combination.

Partially Meeting the Benchmark

Some students understand the structure of the problem but make computational errors. For example, they may incorrectly combine the products of the multiplication combinations they used to derive the product of 6 × 9. After you review their assessments, ask these students to double-check their work. For example: "I see that you used 4 × 9 = 36 and 2 × 9 = 18 to find the product of 6 × 9. Your paper says that 36 + 18 = 53. How could you double-check that?"

Encourage these students to take their time and work carefully to avoid such errors in the future. In addition, note whether these kinds of errors are consistent across problems or more of a one-time occurrence.

Not Meeting the Benchmark

These students do not accurately find the product of 6 × 9 and fail to use known combinations to help them. Ask questions to help them think about related combinations they might know. For example: "Can you tell me what 2 × 9 equals? Can this help you find the answer to 6 × 9?"

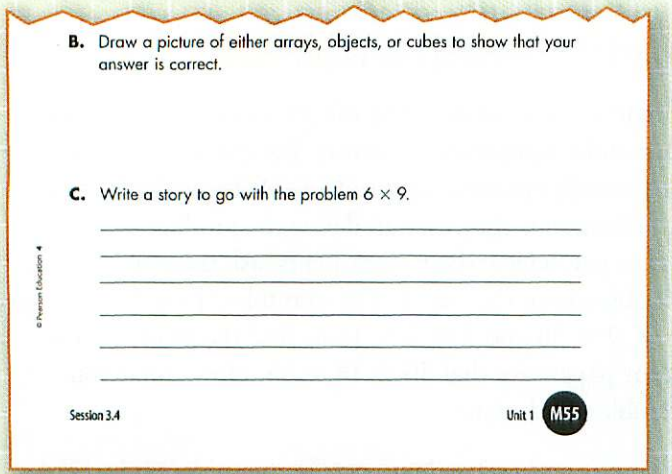
Problem 1 Parts B and C

Benchmark addressed:

Benchmark 2: Use arrays, pictures or models of groups, and story contexts to represent multiplication situations.

In order to meet the benchmark, students' work should show that they can:

- Draw a representation of 6 × 9 that demonstrates their understanding of this expression as representing 6 groups of 9 or 9 groups of 6 and shows that the product of this combination is 54;
- Interpret the task as writing a story problem involving 6 groups of 9 or 9 groups of 6, including a question or a statement about the product.



B. Draw a picture of either arrays, objects, or cubes to show that your answer is correct.

C. Write a story to go with the problem 6 × 9.

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Session 3.4

Unit 1 M55

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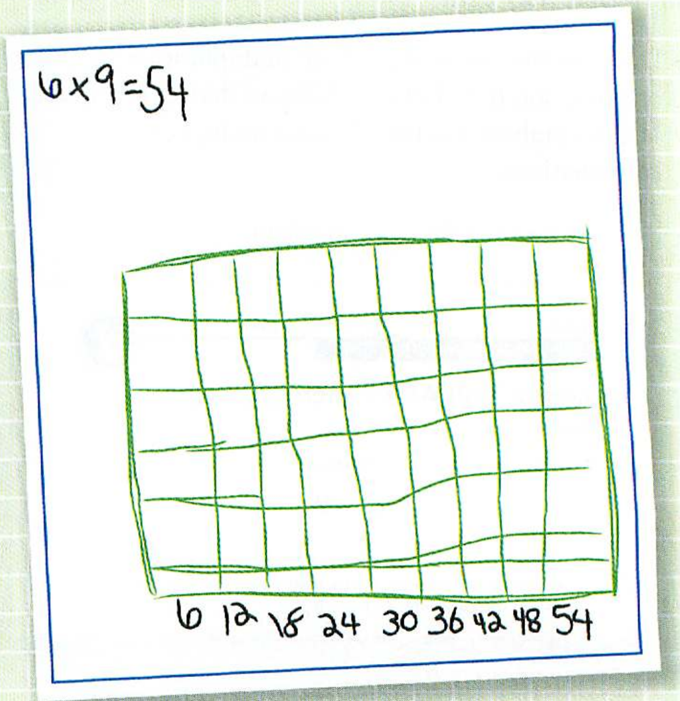
Meeting the Benchmark

The following examples of student work provide a range of typical responses. Both of these students met the benchmark—they were able to represent the meaning of the expression 6 × 9 and show why the product of this combination is 54. They were also able to write story problems involving 6 groups of 9 or 9 groups of 6 and include a question or statement about the product of 6 × 9.

Enrique drew a 6 × 9 array and counted by 6 to label each column.

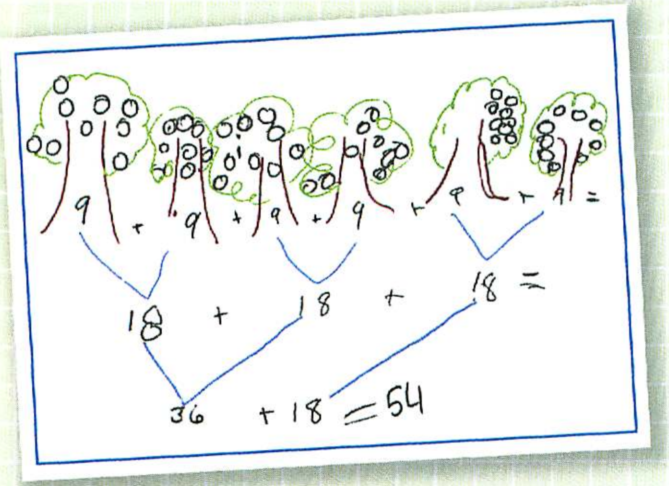


Enrique's Work

He wrote:

"I had 9 packs of juice. In each pack there were 6 boxes. Altogether, I had 54 boxes of juice."

Lucy drew six apple trees with 9 apples on each tree, grouped the 9s to make three 18s, added two of the 18s to make 36, and added the final 18 to make 54.



Lucy's Work

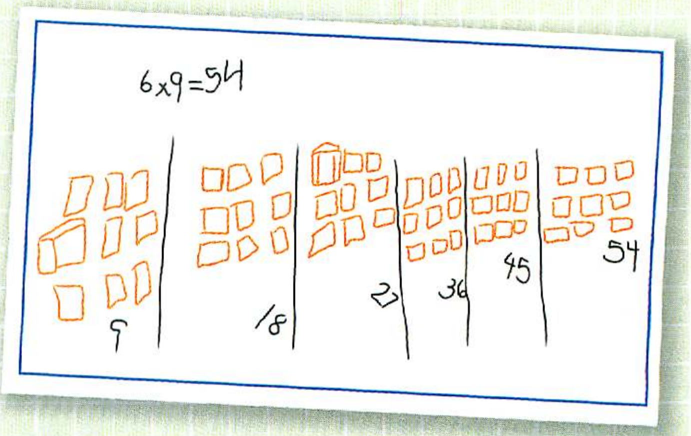
She wrote:

"There are 6 trees. Each tree has nine apples. How many apples are there altogether?

Partially Meeting the Benchmark

These students have completed part of the task successfully. They may have drawn a correct representation of 6 × 9 but failed to show why the product of this combination is 54. Ask these students how they could prove to you that there are 54 objects in their representation. Other students, as in the examples below, may have drawn accurate representations of 6 × 9 but failed to write a complete and/or accurate story problem.

Amelia's representation correctly showed 6 groups of 9 and demonstrated that the product of 6 × 9 is 54. She drew the following representation, showing 6 groups of 9 cubes and skip counted by 9 to show the product.

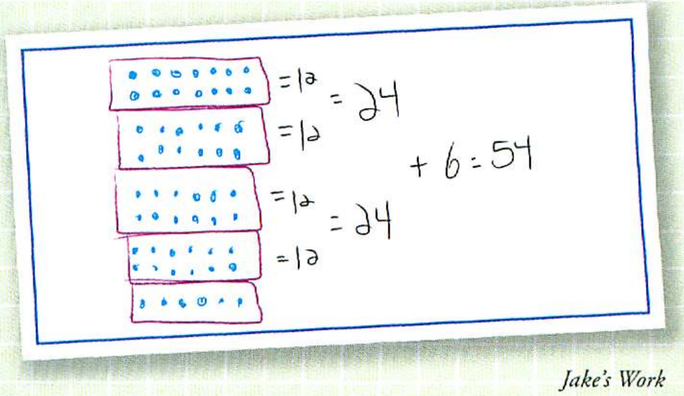


Amelia's Work

However, the story problem she wrote ("There were 6 groups of people. They each had 9 papers. How many people are there?") demonstrates confusion about the meaning of the numbers in the problem.

Ask students such as Amelia to read the story problem they wrote out loud to see whether they self-correct. If they do not see their error, ask questions to help them focus on the meaning of each number. Ask them to draw a picture showing what they mean. For example: "What does the 6 mean in your story? Show me a picture of what you mean. Tell me in your own words what is happening in your story."

Jake drew a 9 × 6 array of dots and grouped the dots into 4 groups of 12 and 1 group of 6 as seen below.



Jake's Work

Jake found the correct product, but he wrote:

"There are 6 beds: 9 people in each bed."

Ask students such as Jake whether they think that the story problem is complete. For example: "I see you wrote about 6 beds with 9 people in each one. What piece of information do we not know about this multiplication situation? What question could you ask about your problem?"

Not Meeting the Benchmark

These students are unsuccessful in making a representation or writing a story problem for 6 × 9 that demonstrates an understanding of the expression as representing 6 groups of 9 or 9 groups of 6. Sketch a representation of a simpler combination for these students, and ask them what equation they could write to describe what you drew. For example: "What do you see in the picture I just drew? What equation could I write to go with this picture?"

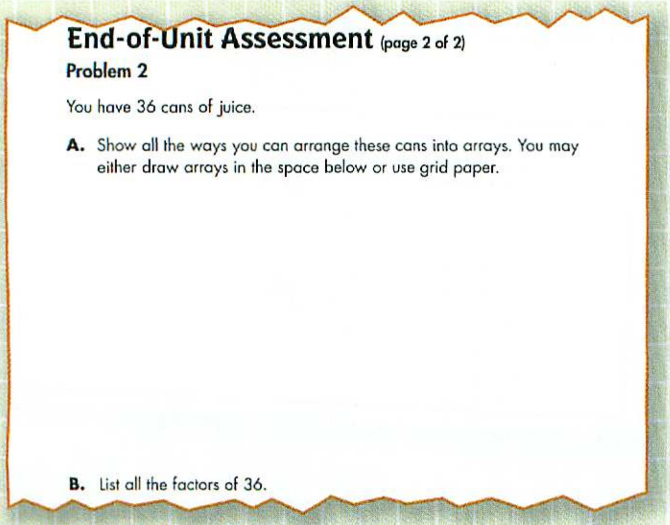
Problem 2

Benchmark addressed:

Benchmark 3: Find the factors of 2-digit numbers.

In order to meet the benchmark, students' work should show that they can:

- Draw all of the possible arrays for the number 36;
- List all of the factors of 36.



End-of-Unit Assessment (page 2 of 2)

Problem 2

You have 36 cans of juice.

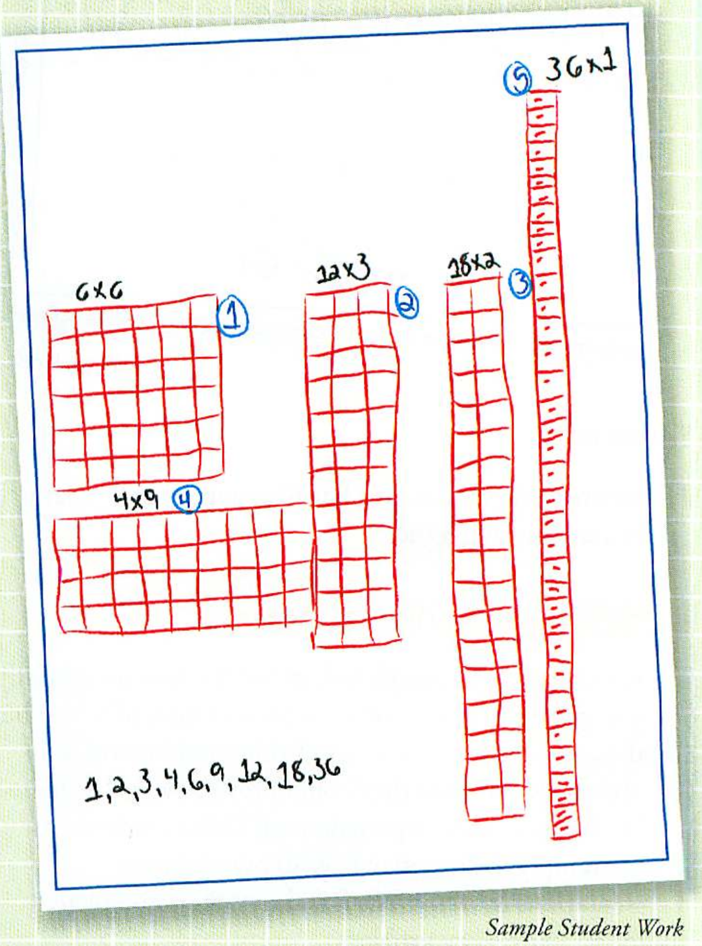
A. Show all the ways you can arrange these cans into arrays. You may either draw arrays in the space below or use grid paper.

B. List all the factors of 36.

▲ Resource Masters, M56

Meeting the Benchmark

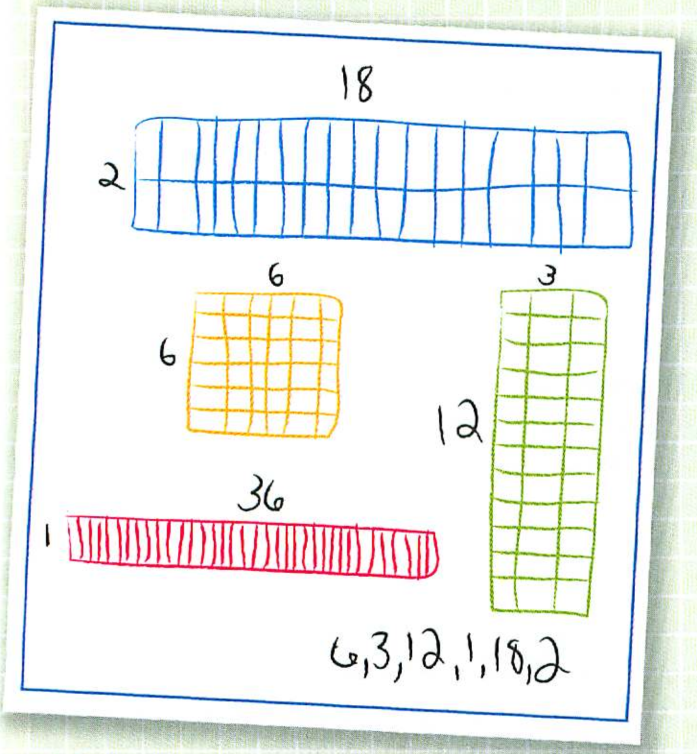
These students tend to approach this task in an organized manner that helps them make sure that all of the factors have been listed (e.g., writing the factors in pairs before making the list and then writing the factors in order from either largest to smallest or from smallest to largest).



Sample Student Work

Partially Meeting the Benchmark

These students successfully construct most but not all of the arrays for 36 and list most but not all of the factors. They tend to be less organized in their approach to this task, not listing the factors in pairs or in order. For example, they may list 1 as a factor of 36 but fail to include 36 as a factor as well.



Sample Student Work

Ask these students whether they think they have made *all* the possible arrays for this number and listed all of the factors. Focus their attention on factor pairs. For example: "You listed 1 as a factor of 36. If 1 is a factor, what other number must be a factor as well? 1 times what equals 36?"

Not Meeting the Benchmark

Students who make no arrays or only a couple of arrays for this number may need more practice in making arrays for numbers smaller than 36. Ask these students to make arrays for the number 12, and help them think about the pairs of factors for this number and how to put these factors in order.

Dialogue Box

How Many in This Array?

After solving the problem on *Student Activity Book* page 1, *How Many in This Array?*, students are discussing how they determined the number of cans in the open case of juice.

Richard: Sabrina and I solved it by counting by 6, because there are 6 cans going down.

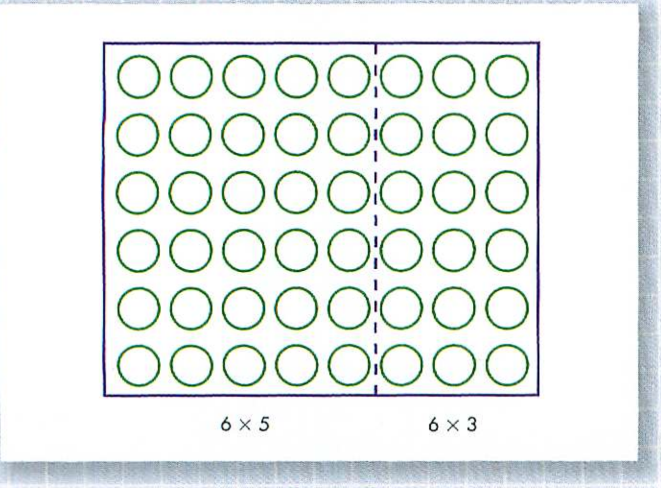
Sabrina: Yeah, it's a 6 by 8 array. There are 8 cans on the top and 6 going down.

Alejandro: I did almost the same thing as Sabrina and Richard. I did 6 × 8 and I skip counted 6, 12, 18, [class joins him] 24, 30, 36, 42, 48.

Marisol: You know how she said that you can count by 6. Well, you can count by 5, too.

Teacher: Marisol has noticed the 6 unobstructed rows of 5 that can be seen on the left side of the array.

Marisol: I counted 5 across and then stopped. I did 5 by 6. 5 × 6 is 30. Then what's left is 3 more in each row and there are 6 rows.



Teacher: So what's 6 × 3?

Marisol struggles a bit, but with some help from her classmates says, "18."

Teacher: So you did 6 × 5 and got 30 and 6 × 3 and got 18. How many cans are in the whole array?

Steve: 30 plus 18 equals 48. There are 48 cans in the array.

Richard: That's the same answer that Sabrina and I got when we counted by 6 eight times.

The teacher records: $(6 \times 5) + (6 \times 3) = 6 \times 8$

$30 + 18 = 48$

Teacher: So there are two different ways that we've solved this problem so far. Sabrina, Alejandro, and Richard saw 6 rows of 8 and thought of the problem as 6 × 8. Marisol saw 6 rows of 5, or 6 × 5, and 6 rows of 3, or 6 × 3, and then put those together. All of them got 48 as their answer.

Breaking apart a more difficult multiplication combination such as 6 × 8 into two simpler combinations such as (6 × 5) + (6 × 3) is an important strategy that students will need as they move into multidigit multiplication problems. Through recording Marisol's strategy for determining the number of cans as an equation equaling 6 × 8, this teacher helped her students see the connection between Marisol's strategy and Richard and Sabrina's strategy.

Dialogue Box

Another Array Picture

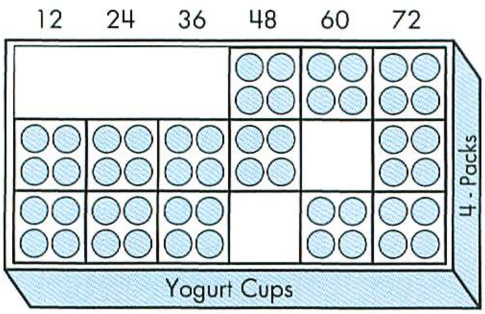
After solving the problem on *Student Activity Book* page 9, *Another Array Picture,* students are discussing how they determined the number of yogurts that would be in the case if it were full.

Amelia: I counted by 12s.

Teacher: Where do you see 12s in this picture?

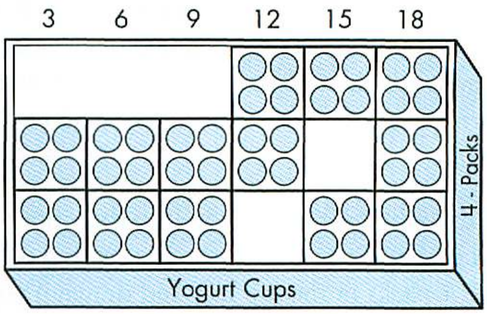
Amelia points to the full column of three 4-packs on the right side of the picture.

Amelia: There's 12 in each column, because it's 3 times 4. And I can count by twelves: 12, 24, 36, 48, 60, and 72.



12 × 6 = 72

Bill: I knew that there were 3 there [pointing to the same column of three 4-packs], so I counted by 3 and got 18. [Bill counts the columns in the same way that Amelia did, only by 3s instead of 12s.] Then I multiplied by 4 because there were 4 in each pack. I did 18 times 2 equals 36, and then I doubled it and got 72.



18 × 4 = 72

Teacher: Why can Amelia count by 12s and Bill can count by 3s, and they both come up with 72 as the answer?

Yuson: Because Amelia counted the yogurts, and Bill counted the packs.

Luke: It's like they're both counting by 3s, because 12 has 3s in it.

Jill: Yeah, so 3 is a factor of 12, if you can count something by 12s, you can count by 3s.

These students are beginning to make sense of an idea that they will be investigating further in this unit, which is that if one number is a factor of another number, it is also a factor of the multiples of that other number. For now, the teacher acknowledges this and moves on to collect other strategies for solving the problem.

Enrique: I did 12 × 6, because it's 12 in each column and there are 6 of them. I knew two 12s were 24, then I added 24 plus 24 plus 24.

Cheyenne: I drew 4 in each of the empty places. [She has filled in the missing 4-packs of yogurt so that the array is complete.] I counted them by 5s because that was easier. I got to 90, then I counted all the little packs of 4, and there were 18. So I minused 18 and got 72.

Teacher: Can you explain why you counted by 5s instead of 4s, and why you knew you had to subtract 18 to get your answer?

Cheyenne: I knew there were 4 yogurts in each space, but I don't like counting by 4s. I counted by 5s instead. I knew I added 1 more yogurt each time, so I counted the spaces with yogurt and there were 18. I had to minus 1 yogurt for each space.

Steve: I moved two from the top into the empty spaces, and that made 12 times 4. [Steve has crossed out two of the three 4-packs in the top row and drawn 4 yogurts in each of the empty spaces in the second and third row, completing these horizontal rows.] That's 48, then 4 more from the pack left at the top is 52. Then I added 5 empty packs; 5 times 4 is 20, and 52 plus 20 is 72.

In each of these strategies, students are using multiplication combinations that they either already know or can access easily by skip counting. They are considering the array as a whole, despite its missing pieces, and breaking it apart into manageable pieces. All of these are key strategies for solving multiplication problems with larger numbers.

Strategies for Learning Hard Combinations

This class is discussing how to use multiplication combinations they know to find the product of more difficult combinations. The teacher writes 3 × 3 and 3 × 6 on the board.

Teacher: Look at these two problems, 3 × 3 and 3 × 6. You probably know the product of 3 × 3. In fact, you probably know the answer to both of these, but we want to look at the way problems can have connections to each other. We know that 3 × 3 is 9, but how does knowing one math problem help you with solving the other one? If 3 × 3 is 9, how does that help you find the answer to 3 × 6?

Derek: 3 × 3 and 3 × 3 is the same as 3 × 6.

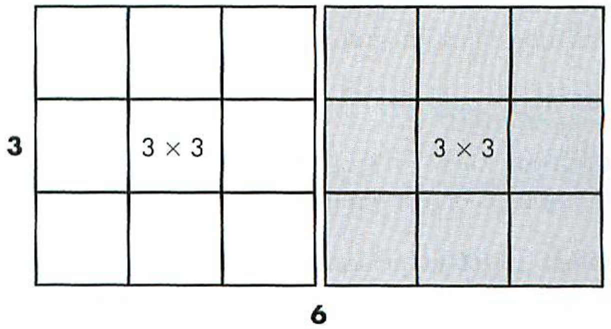
The teacher writes (3 × 3) + (3 × 3) on the board.

Derek: You just combine the two to get 18. 3 × 3 plus 3 × 3 equals 18.

Teacher: So Derek thinks that 3 × 3 plus another 3 × 3 is the same as 3 × 6. Who can suggest a representation of some sort or a story problem that would show that this is true?

Jill: I have a way. You could do an array for 3 × 3 and then just add another 3 × 3 array next to it. Then instead of 3 rows of 3, you have 3 rows of 6. You can count the rows.

The teacher draws two 3 × 3 arrays next to each other on the board as Jill explains. She also places a 3 × 3 Array Card and 3 × 6 Array Card on the overhead to demonstrate the doubling of 3 × 3 to get 3 × 6.



Teacher: Who understands what Jill is saying about the arrays?

Amelia: You can use 3 × 3 and get 9, and all you have to do is add another 9 on it.

Lucy: You're doubling it.

Teacher: What about a story problem? Who could tell a multiplication story that would help us see how 3 × 3 can help with figuring out 3 × 6? . . .

There's quite a long silence, so the teacher then makes a suggestion.

Teacher: Remember when we had the problem about 3-packs of juice? What if we wanted to know how many juice boxes are in six 3-packs of juice? How could knowing 3 × 3 help?

Richard: You know that 3 × 3 is 9, so that's 3 of the packs. But you still have 3 more packs, so, that's 9, and then, um . . . [Richard trails off and seems to have lost track.]

Teacher: How about if I draw this on the board? So here's three 3-packs of juice, and Richard says there are 9 juice boxes in three 3-packs. But we need six 3-packs. So, without drawing the rest, who can continue what Richard started?

Cheyenne: It's just 3 more. It's the same. It's 3 and 3 again. So you have 9, so you'd have the same thing again, another 9, that makes 18.

Students move on to discuss what multiplication combinations they know that can help them solve 8 × 7.

Nadeem: I would do 7 × 7 then plus 7.

Jake: If you knew 10 × 7, which is 70, you could do that and minus 7s or just minus 14.

Teacher: That's an interesting one. Jake says you could start with 10 × 7, which is too big, and then subtract 7s. Who can come up with a story that would help us with this one? You could use the juice cans again or something else.

Alejandro: I know. It's like if you have 8 tables in the cafeteria, and there are 7 kids sitting at each table, and you want to know how many kids. If you had 10 tables, that would be 70 kids, but you have 2 tables less, so you have to take away two of the tables. So 70 minus 7, minus 7.

Anna: You could also do 8 × 5 and then add 8s to it.

Noemi: You could skip count and start at 8 × 2 or start from 0 . . .

Anna: But you could start at 8 × 5 and keep skip counting. That would be quicker.

Teacher: What about doubling? Would that help with this problem? What could you start with?

Nadeem: 4 × 7 is 28.

Jill: It's like what I did before with 3 × 3 and 3 × 6. You could draw a 4 × 7 array and add another 4 × 7 array next to it.

These students are effectively using multiplication combinations they know to solve more difficult problems and are on their way to developing fluency with the basic multiplication combinations. See **Teacher Note:** Learning and Assessing Multiplication Combinations on page 120.

The teacher wants students to get into the habit of using representations and stories to refer to as they reason about multiplication. Even though many of the students in her class can reason about these relatively small numbers without referring to pictures or story problems, she knows that there are some students who need to ground their thinking in such representations.

She also knows that later this year, as students work with breaking apart multiplication problems with larger numbers, many of them will need to have visual images to help them think through which parts of the problem they have solved and which parts they still need to do. By beginning to develop the habit now of creating a picture, diagram, or story, they are learning about important tools to help them reason about multiplication.

Identifying Factors and Multiples in *Multiple Turn Over*

Students have just finished playing one game of *Multiple Turn Over.* Before playing a second time, they discuss with the teacher their strategies for determining what factors to choose and which cards to turn over after a factor has been named. They have their recording sheets in front of them.

Ramona: I chose factors that I know would let me turn over a lot of cards. Like 5. I knew I could turn over a lot.

Teacher: So tell me something about 5 that helped you turn over a lot of cards.

Ramona: I had the Multiple Card 65.

Teacher: How did you know that 5 is a factor of 65?

Ramona: I know that 5 is a factor of 60 and if you count by 5 from it, you get 65. I know that if I count by 5, I land on all the multiples of 10 and when I get to 60, I just add 5.

Enrique: For one of my turns, I chose the factor 2, because I had numbers that end in 4, 2, 0, 6, and 8. I knew I would land on those numbers when I count by 2.

Teacher: What do you know about those numbers that you will land on when you count by 2?

Helena: Those are the even numbers.

Derek: I tried not to use 2 as a factor, because then I knew that my partner could turn over a lot of cards, too.

Teacher: So you thought about other factors for the even numbers on your Multiple Cards that you could use instead of 2?

Derek: Yeah. I had 12 and 36 and 60, so I used the factor 6 instead.

Lucy: I had 87 and I thought that 3 might be a factor.

Teacher: How did you decide whether you were right about that?

Lucy: Steve was my partner, and we decided to count by 3 on a number line. We landed on 87, so we knew it was a multiple of 3.

Teacher: Where did you start when you counted on the number line?

Lucy: We started at nine, because we know that 9 is a multiple of 3. We didn't want to start at the very beginning.

Teacher: Could you have started at a higher multiple of 3? How could knowing 10 × 3 help you?

Steve: Oh! We could have started at 30 because ten 3s would get us to 30.

The teacher writes $10 \times 3 = 30$ and $20 \times 3 =$ ____ on the board.

Teacher: So if you know that ten 3s will get you to 30, how far will twenty 3s get you? How far away from 87 would you be?

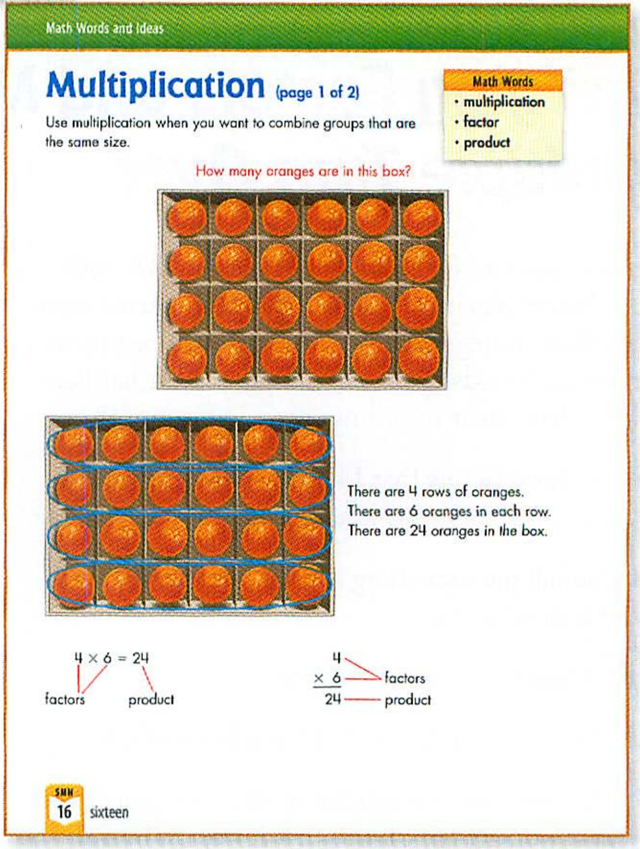
Lucy: We would get to 60. That's 27 away—nine more 3s.

The teacher in this classroom helped her students consider how to use known multiplication relationships to identify factors and multiples while playing *Multiple Turn Over.* Although Lucy and Steve's strategy of counting by 3s did result in correctly determining that 3 is a factor of 87, the teacher's questions encouraged them to use known multiplication combinations such as 10 × 3 and 20 × 3 to solve the problem more efficiently. Starting with a large "chunk" of the problem is a strategy that will serve students well as they begin multiplying and dividing with larger numbers.

Student Math Handbook

The *Student Math Handbook* pages related to this unit are pictured on the following pages. Encourage students to use the **Math Words and Ideas** pages as a summary of the math content covered in class. Remind students to think about and answer the question(s) at the bottom of many of these pages. Students can use the **Games** pages to review game directions during class or at home.

When students take the *Student Math Handbook* home, they and their families can discuss these pages together to reinforce or enhance students' understanding of the mathematical concepts and games in this unit.



Math Words and Ideas

Multiplication (page 1 of 2)

Math Words
- multiplication
- factor
- product

Use multiplication when you want to combine groups that are the same size.

How many oranges are in this box?

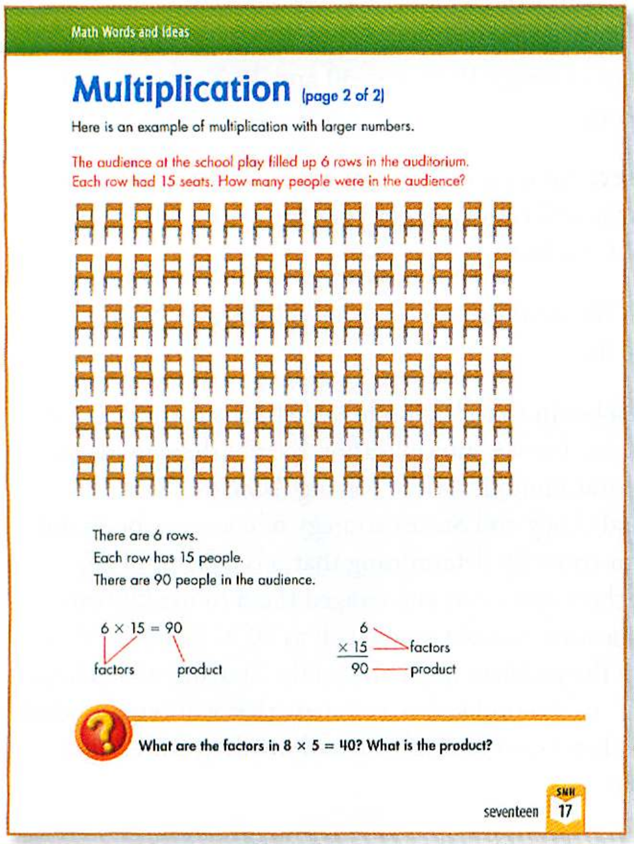
There are 4 rows of oranges.
There are 6 oranges in each row.
There are 24 oranges in the box.

$4 \times 6 = 24$ — factors, product

$\begin{array}{r} 4 \\ \times\ 6 \\ \hline 24 \end{array}$ — factors, product

SMH 16 sixteen

Math Words and Ideas, p. 16



Math Words and Ideas

Multiplication (page 2 of 2)

Here is an example of multiplication with larger numbers.

The audience at the school play filled up 6 rows in the auditorium. Each row had 15 seats. How many people were in the audience?

There are 6 rows.
Each row has 15 people.
There are 90 people in the audience.

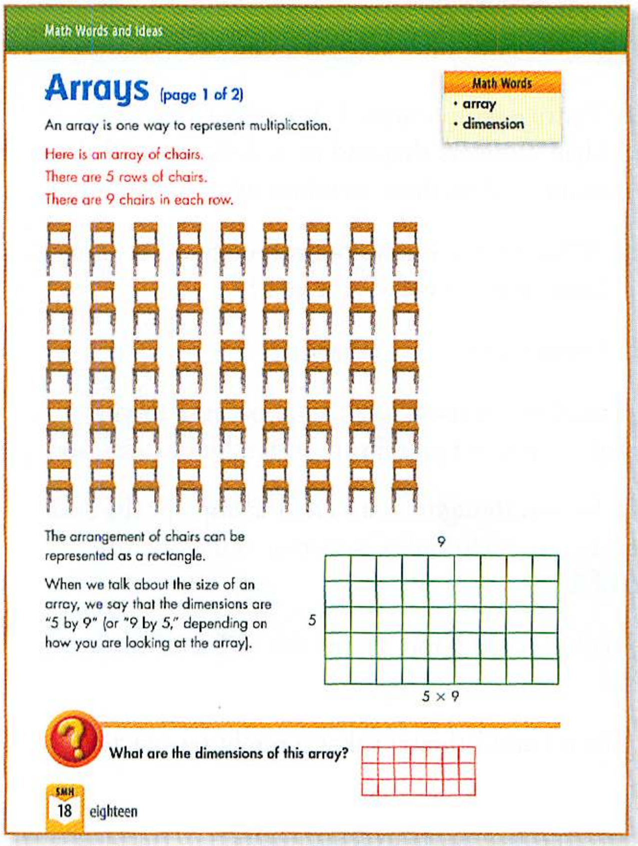
$6 \times 15 = 90$ — factors, product

$\begin{array}{r} 6 \\ \times\ 15 \\ \hline 90 \end{array}$ — factors, product

What are the factors in 8 × 5 = 40? What is the product?

seventeen SMH 17

Math Words and Ideas, p. 17



Math Words and Ideas

Arrays (page 1 of 2)

Math Words
- array
- dimension

An array is one way to represent multiplication.

Here is an array of chairs.
There are 5 rows of chairs.
There are 9 chairs in each row.

The arrangement of chairs can be represented as a rectangle.

When we talk about the size of an array, we say that the dimensions are "5 by 9" (or "9 by 5," depending on how you are looking at the array).

9

5

5 × 9

What are the dimensions of this array?

SMH 18 eighteen

Math Words and Ideas, p. 18