Teacher Note

Learning and Assessing Multiplication Combinations

Students began working on the multiplication combinations in Grade 3, learning combinations with products to 50. They continued on to the rest of the combinations (to 12×12) in the first Grade 4 unit, *Factors, Arrays, and Multiples.* In this unit, students practice and review multiplication combinations to 12×12 as homework, making use of these combinations as they solve multiplication problems in class. The goal is for all students to become fluent with multiplication combinations by the end of this unit. To meet this goal, however, some students will need additional practice during the unit, and some will need continuing practice after the unit is completed. This Teacher Note provides recommendations for supporting students in this ongoing practice.

Why Do We Call Them *Combinations*?

The pairs of factors from 1×1 through 12×12 are traditionally referred to as "multiplication facts"—those multiplication combinations with which students are expected to be fluent. The *Investigations* curriculum follows the National Council of Teachers of Mathematics (NCTM) convention of calling these expressions *combinations* rather than *facts. Investigations* does this for two reasons. First, naming *only* particular addition and multiplication combinations as *facts* seems to give them elevated status, more important than other critical parts of mathematics. In addition, the word *fact* implies that something cannot be learned through reasoning. For example, it is a fact that the first president of the United States was George Washington, and it is a fact that Rosa Parks was born in Alabama in 1913. If these facts are important for us to know, we can remember them or use reference materials to look them up. However, the product for the multiplication combination 6×7 can be determined in many ways; it is logically connected to our system of numbers and operations. If we forget the product, but understand what multiplication is and know some related multiplication combinations, we can find the product through reasoning. For example, if we know that $5 \times 7 = 35$, we can add one more 7 to determine that the product of 6×7 is 42. If we know that $3 \times 7 = 21$, we can reason that the product of 6×7 would be twice that, $2 \times (3 \times 7) = 42$.

The term *facts* does convey a meaning that is generally understood by some students and family members, so you will need to decide whether to use the term *facts* along with *combinations* in certain settings in order to make your meaning clear.

Fluency with Multiplication Combinations

Like NCTM, this curriculum supports the importance of students learning the basic combinations through a focus on reasoning about number relationships: "Fluency with whole-number computation depends, in large part, on fluency with basic number combinations—the single digit addition and multiplication pairs and their counterparts for subtraction and division. Fluency with basic number combinations develops from well-understood meanings for the four operations and from a focus on thinking strategies. . . ." (*Principles and Standards for School Mathematics*, pages 152–153)

Fluency means that combinations are quickly accessible mentally, either because they are immediately known or because the calculation that is used is so effortless as to be essentially automatic (in the way that some adults quickly derive one combination from another).

Helping Students Learn the Multiplication Combinations

A. Students Who Know Their Combinations to 50

Students who know their combinations to 50, as well as the combinations that involve multiplying by 10 up to 100 (6×10, 7×10, 8×10, 9×10, 10×10), can work on learning the most difficult combinations. Here is one way of sequencing this work.

1. Learning the remaining combinations with products to 100. There are 6 difficult facts to learn (other than the $\times 11$ and $\times 12$ combinations, which are, in fact, not as difficult as these, and are discussed below). These six difficult combinations are: 6×9 (and 9×6), 7×8 (and 8×7), 7×9 (and 9×7), 8×8, 8×9 (and 9×8), and 9×9. Note that knowing that multiplication is commutative is crucial for learning all the multiplication combinations. The work with Array Cards supports this understanding, see **Teacher Note:** Representing Multiplication with Arrays, page 117.

Students can work on one or two of these most difficult multiplication combinations each week. Make sure that they use combinations they do know to help them learn ones they don't know—for example, $8 \times 7 = 2 \times (4 \times 7)$, or $9 \times 7 = (10 \times 7) - 7$. They can write these related multiplication combinations as "start with" hints on the Multiplication Cards. If most of your class needs to work on the same few hard combinations, you might want to assign the whole class to focus on two of these each week.

2. Learning the ×11 and ×12 combinations. We consider these combinations to be in a different category. Historically, these combinations were included in the list of "multiplication facts." However, when we are dealing with 2-digit numbers in multiplication, an efficient way to solve them is through applying the distributive property, breaking the numbers apart by place as you would with any other 2-digit numbers. We include them here because some local or state frameworks still require knowing multiplication combinations through 12×12. In addition, 12 is a number that occurs often in our culture, and it is useful to know the $\times 12$ combinations fluently. Most students learn the $\times 11$ combinations easily because of the pattern (11, 22, 33, 44, 55, . . .) created by multiplying successive whole numbers by 11. They should also think through why this pattern occurs: $3 \times 11 = (3 \times 10) + (3 \times 1) = 30 + 3 = 33$. They should think through why $11 \times 10 = 110$ and $11 \times 11 = 121$ by breaking up the numbers. Students can learn the $\times 12$ combinations by breaking the 12 into a 10 and 2, e.g., $12 \times 6 = (10 \times 6) + (2 \times 6)$. Some students may also want to use doubling or adding on to known combinations: $12 \times 6 = 2 \times (6 \times 6)$, or $12 \times 6 = (11 \times 6) + 6$.

B. Students Who Need Review and Practice of Combinations to 50

Students who have difficulty learning the multiplication combinations often view this task as overwhelming—an endless mass of combinations with no order and reason. Bringing order and reason to students' learning of these combinations in a way that lets them have control over their progress is essential. Traditionally, students learned one "table" at a time (e.g., first the $\times 2$ combinations, then the $\times 3$ combinations, the $\times 4$ combinations, and so on). However, the multiplication combinations can be grouped in other ways to support learning related combinations.

First, make sure that students know all multiplication combinations that involve $\times 0$, $\times 1$, $\times 2$, $\times 5$, and $\times 10$ (up to 10×10) fluently. (Students worked with the $\times 0$ combinations in Grade 3.) Note that, although most fourth graders can easily count by 2, 5, and 10, the student who is fluent does not need to skip count to determine the product of multiplication combinations involving these numbers.

When students know these combinations, turn to those that they have not yet learned. Provide a sequence of small groups of combinations that students can relate to what they already know. There are a number of ways to do this.

1. Learning the ×4 combinations. Work on the ×4 combinations that students do not yet know: 3 × 4, 4 × 4, 6 × 4, 7 × 4, 8 × 4, and 9 × 4. Help students think of these as doubling the ×2 combinations. So, 4 × 6 = (2 × 6) + (2 × 6), or 4 × 6 = 2 × (2 × 6). Students may verbalize this idea as "4 times 6 is 2 times 6 and another 2 times 6," or "to get 4 times 6, I double 2 × 6." Doubling is also useful within the ×4 combinations; for example, when students know that 3 × 4 = 12, then that fact can be used to solve 6 × 4: 6 × 4 = (3 × 4) + (3 × 4). Getting used to thinking about doubling with smaller numbers will also prepare students for using this approach with some of the harder combinations.

2. Learning the square numbers. Next, students learn or review the four remaining combinations that produce square numbers less than 50: 3 × 3, 5 × 5, 6 × 6, and 7 × 7. These are often easy for students to remember. If needed, use doubling or a known combination for "start with" clues during practice (e.g., 6 × 6 is double 3 × 6; 5 × 5 is 5 more than 4 × 5). Students can also build these combinations with tiles or draw them on grid paper to see how they can be represented by squares.

3. Learning the remaining combinations with products to 50. Finally, learn or review the six remaining combinations with products to 50: 3 × 6 through 3 × 9, 7 × 6, and 8 × 6. First, relate them to known combinations (e.g., double 3 × 3 or halve 6 × 6 to get 3 × 6), and then practice them.

Assessing Students' Knowledge of Multiplication Combinations

In Investigation 3 of this unit, students are assessed on their fluency with the multiplication combinations. For this assessment, students are expected to be able to solve 30 problems that are representative of the set of combinations to 12 × 12, with accuracy, in three minutes. If they can solve them all within that time limit, students are either accessing these combinations from memory or they are able to make a very quick calculation that is almost automatic. Some students may take longer than others to reach this level of fluency. You can expect to have students in your class who may need to do this assessment more than once. They will continue to identify combinations they still need to work on and will practice those with their Multiplication Cards. These students may also use Array Cards for practice, either by laying them out factor side up and identifying the products or by playing *Missing Factors*. You may have other favorite practice methods or activities that you want to suggest for particular students. Also, enlist parents or other family members to help with this practice.

In this unit, students also begin to work on division problems that are counterparts to the multiplication combinations. They will continue to work on these in the next multiplication and division unit, *How Many Packages? How Many Groups?* In the meantime, as students work on solving division problems, help them relate division expressions to the multiplication combinations they know; for example, what multiplication combination can help you solve 24 ÷ 6?

Fluency Benchmarks for Learning Facts Through the Grades

Addition: Fluent by end of Grade 2, with review and practice in Grade 3

Subtraction: Fluent by end of Grade 3, with review and practice in Grade 4

Multiplication: Fluent with multiplication combinations with products to 50 by the end of Grade 3; up to 12 × 12 by the middle of Grade 4, with continued review and practice

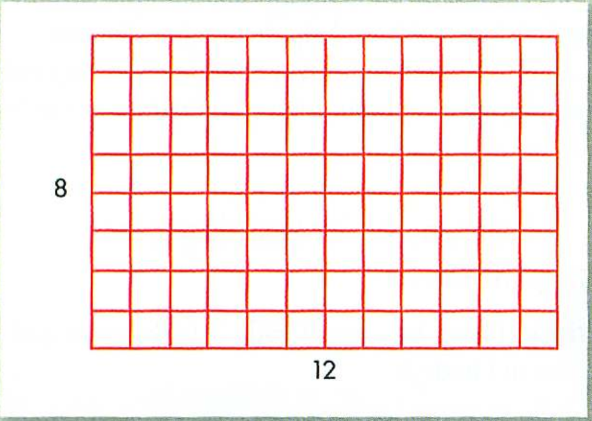
Division: Fluent by end of Grade 5

Teacher Note

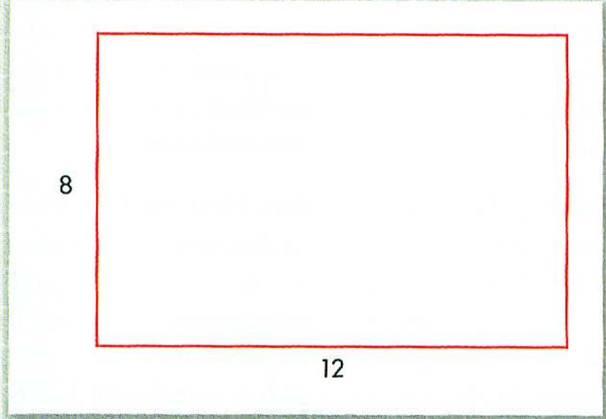
Visualizing Arrays

In order to use rectangular arrays to visualize breaking up a multiplication problem, students must be able to see that the lengths of the sides of the rectangle represent factors, and the area represents the product. Evidence from research and practice indicates that fully understanding this relationship takes time and experience. As adults, we are so familiar with the relationship between the area of a rectangle and the length of its sides that we may not realize that the relationship is not necessarily obvious to students.

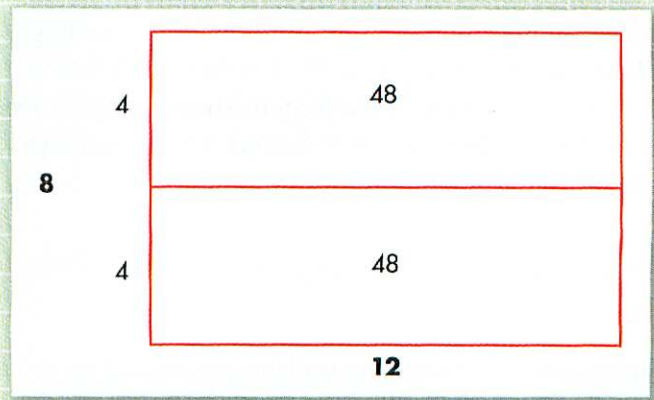
As students are learning to visualize how a rectangular array represents multiplication, they use arrays that show all the individual units of the area. Students can describe the area in terms of the dimensions of the rectangle. For example, in this rectangle, there are 8 rows with 12 squares in each row.

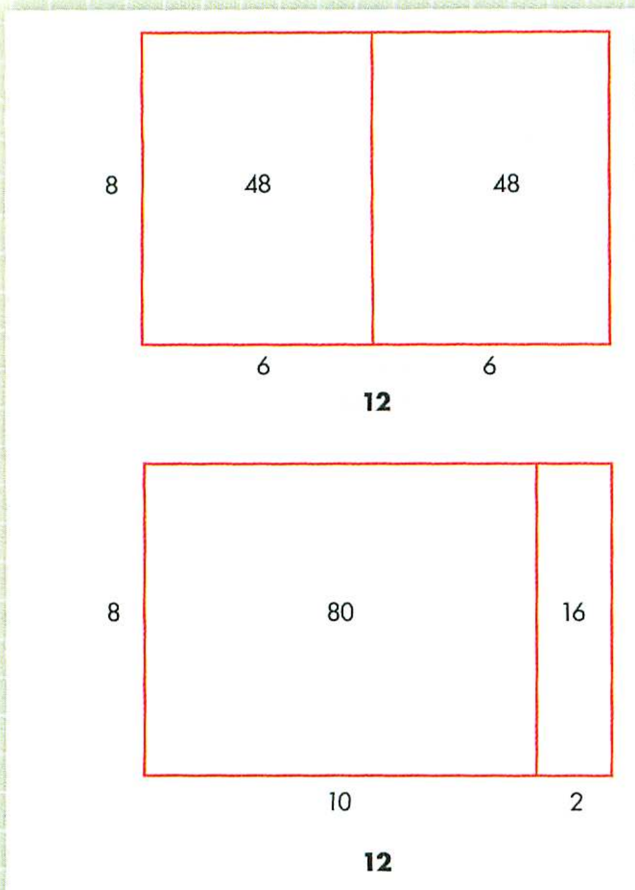


In order to help students visualize a rectangular array to represent any multiplication problem, you will introduce unmarked arrays (arrays without a grid of units) in Investigation 1. Thus, for the problem 8×12, you can draw an unmarked array like this:

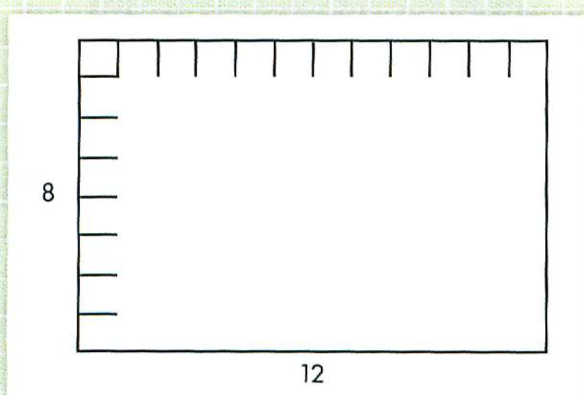


Students can use the unmarked array to help them think through the ways they might break up the array to help them solve the problem.





As a transition between marked and unmarked arrays, you can add tick marks along two sides of the rectangle.



At first, you should present unmarked arrays with sides drawn to fairly closely represent the relative proportion of the numbers in the multiplication expression, as in the first array for 8 × 12 above. Students are learning to use and visualize arrays, so using correct proportions in the diagram helps some students visualize how the array represents multiplication. Eventually, when students are more confident about sketching an unmarked array to help them think through how to solve a problem, you and your students may sketch the arrays without trying to be as accurate about showing the relative length of the sides.

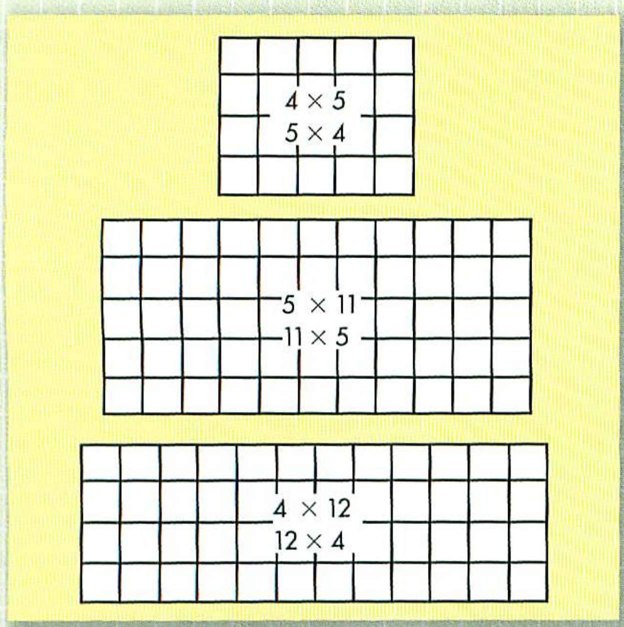
It is essential that students thoroughly understand how an array represents a multiplication (or division) expression before they use an unmarked array to solve problems. For a multiplication expression such as 8 × 12, they must be able to explain where the 8 is in the representation, where the 12 is, and why the area of the rectangle is the product of these two numbers.

Playing *Small Array/Big Array*

The Array Card game *Small Array/Big Array* has a complex set of rules that become familiar through practice. It is important that you play this game yourself ahead of time to ensure that you understand it thoroughly and can help students as they learn to play. See *Small Array/Big Array* (M34–M35) for full directions.

Sample Game

Here's an example of a demonstration game that a teacher and student played to help the class learn the rules. This demonstration version uses only 3 center cards. Here are the center cards at the beginning of the game:



Round 1

Player A (teacher) has a 4 × 7 array among her set of cards. She matches it to the 4 × 12 center card.

Player B (student) has a 3 × 11 array in his set of 10 cards. He matches it to the 5 × 11 center card.

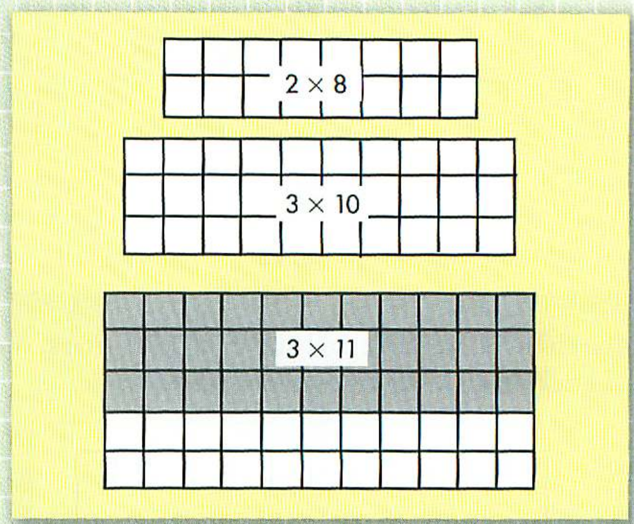
Round 2

Player A doesn't have any more cards with dimensions that match the center cards. She puts the 4 × 5 card from the center onto the 4 × 12 array, completing this match. She collects the match and records it as follows:

$$1.\ 4 \times 12 = (4 \times 7) + (4 \times 5)$$
$$48 = 28 + 20$$

Player A discards one of her cards, 3 × 10, to the center, to replace the 4 × 5 center card that she used. Because she also collected a match on this turn, she draws the top card from the deck, 2 × 8, and replaces the big array that she collected, keeping 3 cards in the center.

Now the cards in the center are

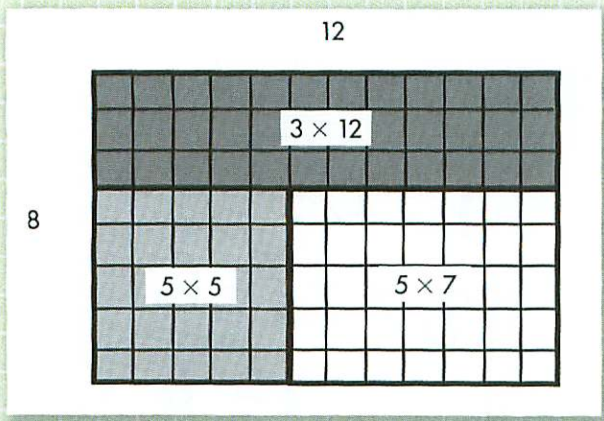


Player B has a 1 × 11 array, which he places on the 5 × 11 array in the center. He recognizes that only a 1 × 11 array can complete the 5 × 11 array. Because there is not another 1 × 11 Array Card in the set, the match is counted as complete. Player B collects this match and records it. The 5 × 11 center card is replaced by a card from the pile.

Common Incorrect Moves

In this game, each small array played on a big array must have one dimension in common with the big array. After one dimension of the big array has been matched, the next array cards played *must match that same dimension.* One way to help students think about this is to say that each new array card must match one of the sides completely.

A common incorrect move occurs when students do not follow this rule. For example, in this game, one player has placed her 3 × 12 array on top of an 8 × 12 array in the center, using the common side length of 12. The next player fits his 5 × 5 array in part of the remaining space, leaving a space that can be filled with a 5 × 7 array.



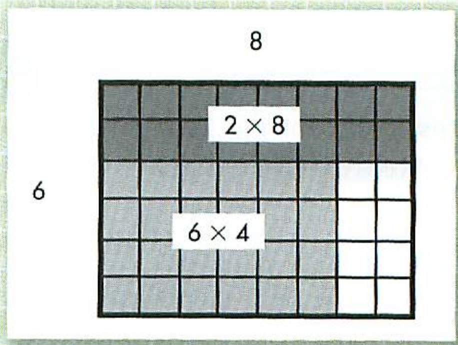
An equation that describes this *Small Array/Big Array* match is

$$8 \times 12 = (3 \times 12) + (5 \times 5) + (5 \times 7)$$

Although the equation is correct mathematically, this arrangement of small arrays is not as helpful for students at this stage. Students need to think through how they can apply the distributive property of multiplication to solve the problem. In later work on more difficult multiplication problems, students will break up both of the numbers in the problem. However, in this unit, breaking up both dimensions can weaken the connection between the array model and students' ideas about multiplication as involving equal groups.

Keeping one dimension common in the big array and its matching small arrays helps students visualize ways to break up one of the numbers in a multiplication problem. For example, if an array has 8 rows with 12 units in each row, students can find the number of squares by first finding how many are in 3 of those groups and then finding how many are in the remaining 5 groups, or 8 groups altogether: $8 \times 12 = (3 \times 12) + (5 \times 12)$. A story context can also be connected to the rectangular array: "I need 8 dozen bagels, but the bagel shop has only 3 dozen left. So I buy those and then go to the supermarket to buy 5 dozen more." Although it is possible to modify this story context so that it can be represented by the more complex equation—$8 \times 12 = (3 \times 12) + (5 \times 5) + (5 \times 7)$—the connection is much more convoluted. For these reasons, it is important to emphasize the rule that one dimension of each small array must match the same dimension of the big array.

Below is another typical incorrect play. In this case, even though the small array 6 × 4 does have one side that matches a dimension of the big array, the card is placed so that the two sides of 6 are not parallel.



It is likely that as they play, students will need reminders about matching one full dimension of the arrays.

Teacher Note

Assessment: Solving 18 × 7

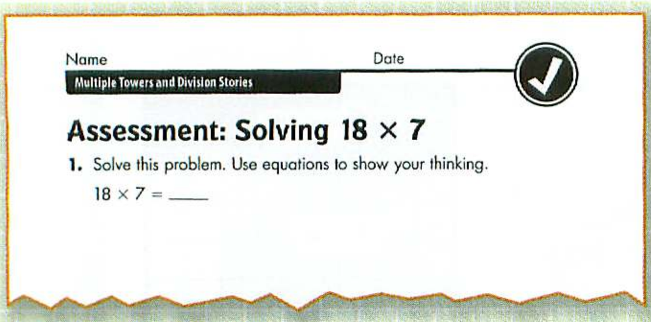
Problems 1 and 2

Benchmark addressed:

Benchmark 1: Multiply 2-digit numbers by 1-digit and small 2-digit numbers (e.g., 12, 15, 20), using strategies that involve breaking the problems apart.

In order to meet the benchmark, students' work should show that they can:

- Break apart the problem in a reasonable way in order to create problems that can be easily solved (see examples that follow).
- Accurately solve each smaller problem.
- Accurately combine the products of each smaller problem.
- Represent their solutions by dividing the array of 18 × 7 and accurately labeling the smaller arrays that are created.

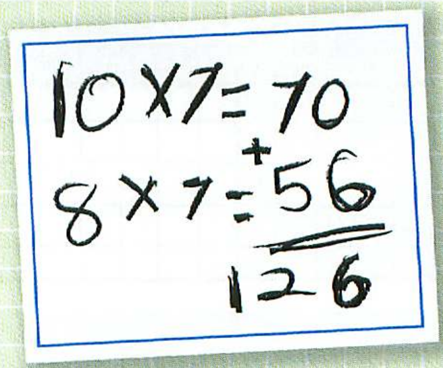

Name Date
Multiple Towers and Division Stories
Assessment: Solving 18 × 7
1. Solve this problem. Use equations to show your thinking.
18 × 7 = ____

▲ Resource Masters, M43

Meeting the Benchmark

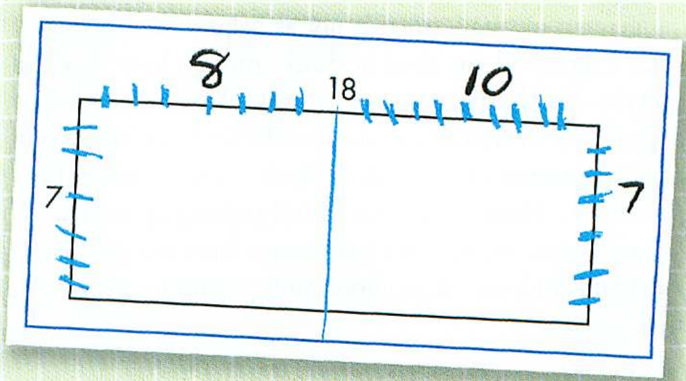
Students who meet the benchmark may break apart the problem in a variety of ways, as shown in the examples that follow. All of these students solve all parts of the problem accurately, combine the parts, and represent their thinking clearly.

Cheyenne broke 18 apart by place value.



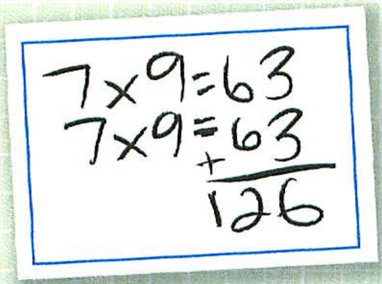
Cheyenne's Work

Yuki broke the problem apart in the same way and drew this array to represent his solution.



Yuki's Work

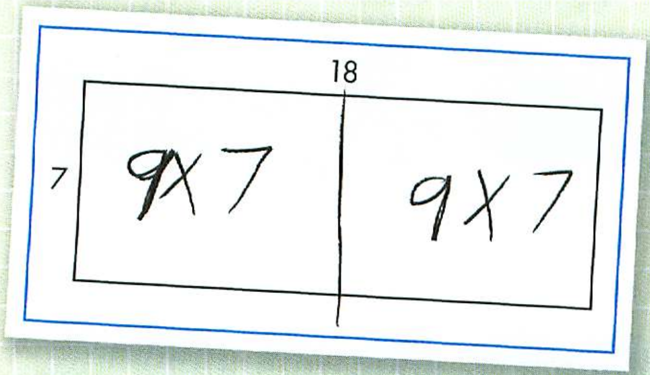
Helena broke 18 in half.



Helena's Work

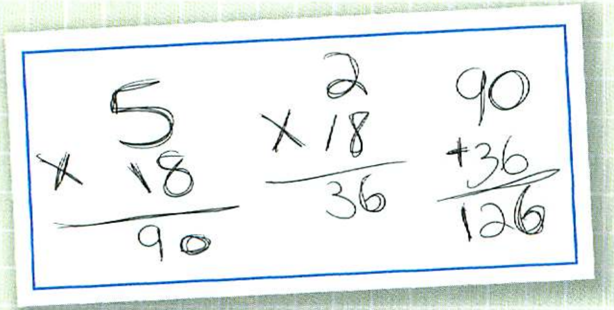
Both Helena and Cheyenne use clear and concise notation to show their work.

This is Nadeem's array for the same strategy Helena used.



Nadeem's Work

Derek knew some multiples of 18. He broke the 7 apart to create two problems that he knew how to solve.



Derek's Work

Partially Meeting the Benchmark

Some students may break apart the problem correctly, but make errors in computation. For example, Anna used Cheyenne's strategy but incorrectly solved 8×7 as 63, probably confusing the product of 8×7 with the product of 9×7. She needs to review any multiplication combinations that she does not know.

Other students may be able to easily and correctly solve the problem but may not be able to represent their solution accurately on the array of 18×7. It is not necessary that students use arrays as their primary means of visualizing multiplication relationships, but they should be able to make sense of this representation. Have these students use Array Cards to construct a variety of array combinations for 18×7, and then ask students how they can match these arrays to different ways of breaking apart the problem.

Not Meeting the Benchmark

Sometimes a student breaks apart a problem in a way that does not help solve it. For example, Bill broke 18×7 into 18×3 and 18×4 but had difficulty solving either of these smaller problems. He seems to know that he should break up one factor of the problem, but does not know why he is doing so or how to follow through and solve the problem. For a student like Bill, check first that the student can interpret the multiplication expression by creating a story context for it. Then determine whether the student can, from that context, think of a multiplication combination related to the problem.

If some students are *not* breaking apart one of the factors in order to solve a problem of this size (for example, they are adding 18 seven times or drawing tally marks and counting), continue to work with them throughout this unit on such activities as *Small Array/Big Array* and on solving multiplication problems by breaking apart one of the factors. Use story contexts to help students identify the part of the problem that has been solved and the part that remains.

Teacher Note

Two Kinds of Division: Sharing and Grouping

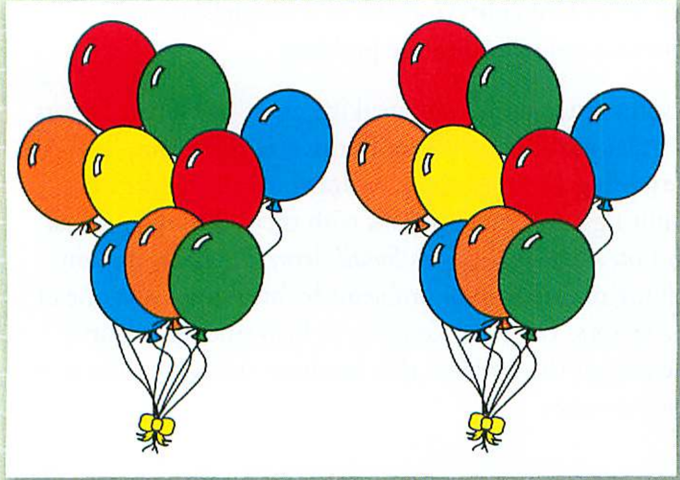
In this unit, students encounter two kinds of division situations. Consider these two problems.

I have 18 balloons for my party. After the party is over, I'm going to divide them evenly between my sister and me. How many balloons will each of us get?

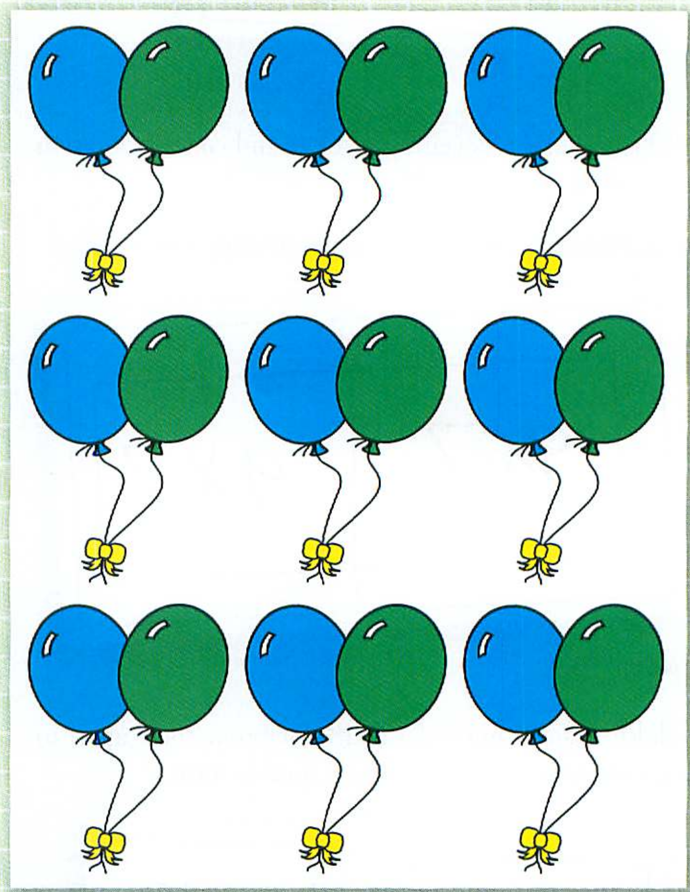
I have 18 balloons for my party. I'm going to tie them together in bunches of 2 to give to my friends. How many bunches can I make?

Each of these problems is a division situation—a quantity is broken up into equal groups. The problem and the solution for each situation can be written in standard notation as $18 \div 2 = 9$. These two situations are actually quite different, however. In the first situation, the number of groups of balloons is given. The question is this: How many balloons will be in each group? In the second situation, the number of balloons in each group is given, and the question is this: How many groups will there be? Each problem involves equal groups of balloons, but the results of the actions look different.

I have 18 balloons and 2 people. How many balloons for each person?



I have 18 balloons to put into bunches of 2. How many bunches?



The solution to each problem is 9, but in the first problem, 9 is the number of balloons per person (the number in each group). In the second problem, 9 is the number of bunches of balloons (the number of groups).

The first situation is probably the one with which your students are most comfortable because it can be solved by dealing out. That is, the action to solve the problem might be as follows: one for you, one for me, one for you, one for me, until all the balloons are given out. In this situation, division is used to describe *sharing*. A more formal term for this kind of problem is *partitive* division—a division situation in which something is distributed and the problem is to determine *how many are in each group.*

In the second situation, the action to solve the problem is making groups—that is, making a group of 2, then another group of 2, and another, and so on, until no balloons are left. In this situation, division is used to describe *grouping.* This situation is sometimes called *measurement division* because the total amount is measured out into equal groups. The formal term for this kind of division situation is *quotative* division—a problem in which the question is *how many equal groups* can be made.

By working with a variety of problems in this unit, students learn to recognize both of these actions as division situations and to develop an understanding that both can be written in the same way: $18 \div 2 = 9$. Depending on the context, help students interpret the notation as either of these questions: If 18 is divided into 2 groups, how many are in each group? or How many groups of 2 are in 18?

As students become more flexible with division, they will understand that they can solve a sharing problem by thinking of it as grouping, or a grouping problem by thinking of it as sharing—whichever way makes it easier to solve. Here is an example:

How many people are on each team if I make 25 equal teams from 100 people?

To solve this problem, it is easy to think, "How many groups of 25 are in 100?" even though the problem is not about groups of 25, but about 25 groups. The numerical answer to this grouping question is also the numerical answer to the sharing problem. Some of your students may soon have an intuitive understanding that they can solve a division problem by thinking about it either way. Students can draw on what they know about multiplication—that 4 teams of 25 is the same number of people as 25 teams of 4 ($4 \times 25 = 25 \times 4$). This understanding is based on the commutative property of multiplication.

Teacher Note

What Do You Do with the Remainders?

Depending on the situation or context of a division problem, remainders affect the solution to the problem in a variety of ways. Consider the following problems involving 44 ÷ 8 and the ways that students look at the remainder in each context.

There are 44 people taking a trip in some small vans. Each van holds 8 people. How many vans do they need?

Andrew: There are 5 vans with 4 people left. You can't leave any people behind, and you can't take half a van, so you'd need 6 vans to take everyone.

If 8 people share 44 crackers equally, how many crackers does each person get?

Ramona: Each person can get 5 crackers. Keep 4 crackers for another day. 44 ÷ 8 = 5 crackers per person with 4 extras.

Steve: Each person can get $5\frac{1}{2}$ crackers. $44 \div 8 = 5\frac{1}{2}$.

If 8 people share 44 balloons equally, how many balloons does each person get?

Venetta: Each person gets 5 balloons. They can't split the four balloons left over, so they will have to decide what to do with them. Maybe they could give them to their teacher. 44 divided by 8 equals 5 with 4 balloons left over.

There are 44 students going to see a movie. Each row holds 8 people. How many rows do they fill up?

Derek: There are 40 people who will fill up 5 rows. Then 4 people have to sit in row 6. 44 divided by 8 equals 5 rows, with 4 more people in another row.

Noemi: You will fill up 5 rows and half of another row. $44 \div 8 = 5\frac{1}{2}$ or 5.5.

On Sunday, 8 friends earned $44 by washing people's cars. They want to share the money equally. How much does each person get?

Emaan: First, each person gets $5. That uses up $40. Then you can split the $4 that's left and give each person $0.50, so each person gets $5.50.

Each of these problems involves dividing 44 by 8. If the problem is presented numerically, the quotient can be written as 5 R4 or $5\frac{1}{2}$ or 5.5. If the problem is given in a context, however, the context determines the answer. Some division problems require whole number solutions, as in the contexts about vans and balloons. Notice that in the van problem, the whole number answer must be greater than the actual numerical quotient, and in the balloon problem, the whole number answer is less than the numerical quotient. In other contexts, a solution can involve fractions or decimals.

By solving problems such as these, students learn to consider the remainder in the context of the problem and to give a solution that makes sense for that problem.

Teacher Note

The Relationship Between Multiplication and Division

Multiplication and division are related operations. For example, here is a set of linked multiplication and division equations.

$8 \times 3 = 24$ $3 \times 8 = 24$

$24 \div 8 = 3$ $24 \div 3 = 8$

After students have become fluent with multiplication combinations to 12×12, they can use these to solve related division equations (sometimes refered to as "division facts") by considering which factor pairs equal a given product. Students often think of a problem such as $24 \div 3 = \_\_\_\_$ as a "missing factor" problem. Using the relationship between multiplication and division, they transform $24 \div 3 = \_\_\_\_$ into $3 \times \_\_\_\_ = 24$. Then they use the multiplication combinations they know to solve the problem: "3 groups of what equal 24? I know that's 3 times 8. So 24 divided by 3 is 8."

The multiplication equations show the multiplication of two factors to equal a product. The division equations show that product divided by one of the factors to equal the other factor. Some problem situations that your students encounter in this unit can be described as both multiplication and division.

For example:

I have a supply of 336 treats for my dog. If I give her 14 treats every week, how many weeks will the supply last?

The quantities in this problem are 336 treats, 14 treats per week, and a number of weeks to be determined. This problem can be written in standard notation as either multiplication or division.

$336 \div 14 = \_\_\_\_$ or $\_\_\_\_ \times 14 = 336$

After the answer to the problem has been found, both division and multiplication equations can be written to show the relationship of the three quantities.

336 treats divided into groups of 14 (14 treats per week) results in 24 groups (weeks):

$$336 \div 14 = 24$$

14 per group (14 treats per week) in 24 groups (weeks) equals 336 treats:

$$24 \times 14 = 336$$

When students solve a problem like this one, they might write either a division equation or a multiplication equation to express the answer and its relationship to the quantities in the problem. Both notations represent the problem, depending on whether the student is thinking of the problem as a division problem or as a multiplication problem with a missing factor. Students should be able to read and interpret both of these notations, explaining what each number in the equation represents and relating the equation to the original problem.

Teacher Note

Assessment: Writing and Solving a Division Problem

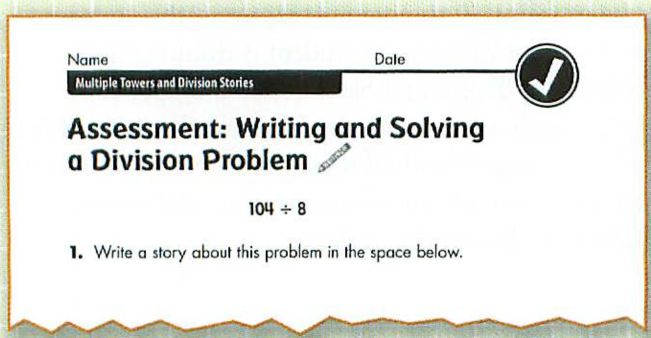
Problem 1

Benchmark addressed:

Benchmark 3: Use story problems, pictures, or concrete models to represent division situations.

In order to meet the benchmark, students' work should show that they can:

- Understand division as splitting a quantity into equal groups;
- Write a story problem in which 104 is divided into groups of 8 or into 8 groups, with a corresponding question about either the number of groups or the number in each group.


Name Date

Multiple Towers and Division Stories

Assessment: Writing and Solving a Division Problem

104 ÷ 8

1. Write a story about this problem in the space below.

▲ Resource Masters, M46

Meeting the Benchmark

At this point in fourth grade, students should be able to represent 104 ÷ 8 as a story problem.

For example, Nadeem wrote:

"There are 104 kids at the school field day. They are making teams of 8 for the games. How many teams do they make?"

Amelia wrote:

"There are 8 girls and they want to share 104 jelly beans. How many jelly beans will each girl get?"

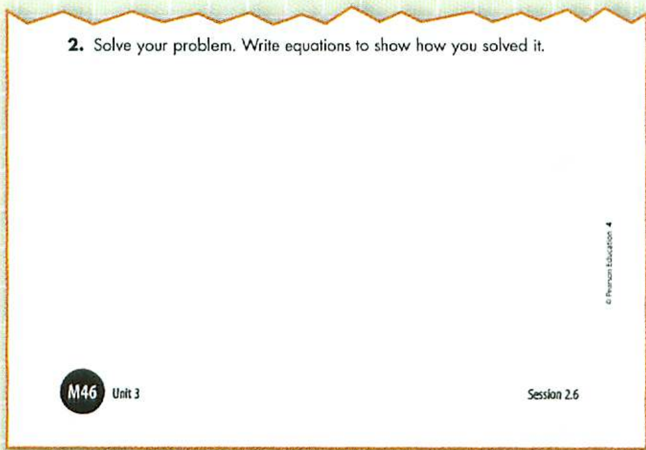
Problem 2

Benchmark addressed:

- **Benchmark 2:** Solve division problems (2-digit and small 3-digit numbers divided by 1-digit numbers), including some that result in a remainder. **Note:** The problem in this assessment does not involve a remainder, but the End-of-Unit Assessment does include a problem with a remainder.

In order to meet the benchmark, students' work should show that they can:

- Accurately solve 104 ÷ 8 and give a correct answer in terms of the problem context;
- Use strategies that involve making groups of the divisor (for example, using known multiplication combinations such as 5 × 8 = 40 or 10 × 8 = 80) or efficiently dividing parts of the dividend (for example, 80 ÷ 8 and 24 ÷ 8).



2. Solve your problem. Write equations to show how you solved it.

M46 Unit 3 Session 2.6

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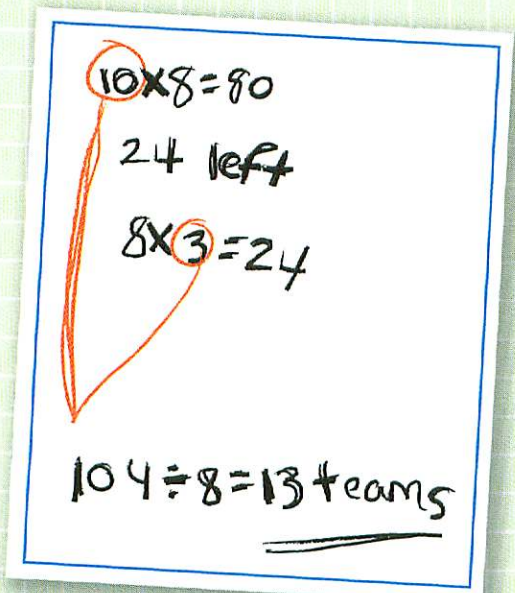
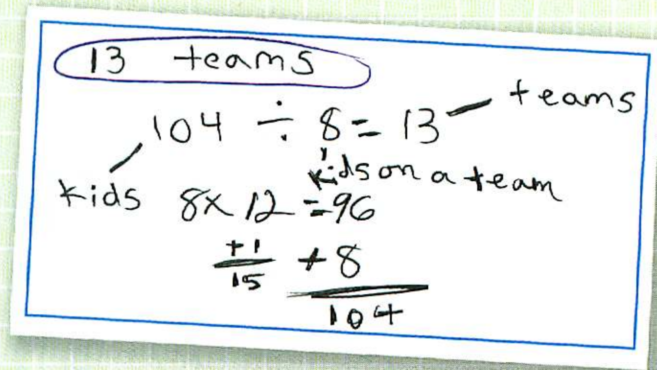
Meeting the Benchmark

Sabrina broke the dividend (104) into 10 groups of 8 plus 3 groups of 8.



Sabrina's Work

Jill used the known combination 12 × 8 to solve this problem.



Jill's Work

Notice how Jill carefully labels each number to correspond with her context.

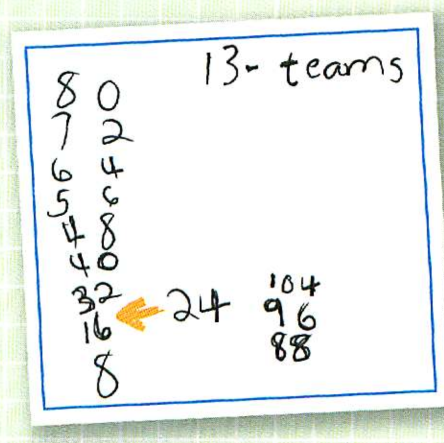
Although using multiplication to solve division problems is stressed in Investigation 2, some students break up the dividend and divide each part by the divisor. For example, Marisol thought of the problem this way:

$$104 \div 8 = (80 \div 8) + (24 \div 8)$$
$$= 10 + 3 = 13$$

Partially Meeting the Benchmark

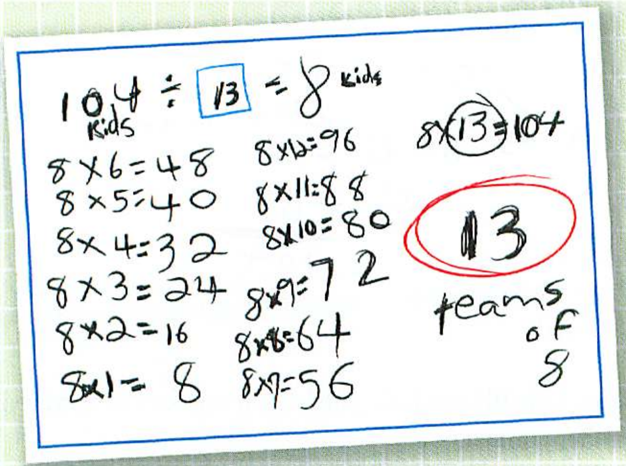
Students who partially meet the benchmark recognize the groups of the divisor in the dividend but solve the problem in a less efficient way, by skip counting or listing all of the combinations involving 8 until they reach 104, for example.

Luke, for example, skip counted to determine the number of 8s in 104.



Luke's Work

Steve listed the multiplication combinations for 8 in order.



Steve's Work

Continue to work with students like these to help them move to a more efficient use of groups of the divisor. For example, tell them: "I see you counted by 8 [or, wrote the combinations involving 8] to figure out the number of 8s in 104. Is there a multiplication combination with 8 as a factor that will get you part of the way there? Do you know 5 × 8 or 10 × 8? How could these combinations help you?"

Some students use efficient strategies but make minor computation errors. Help these students find ways to double-check their work.

Not Meeting the Benchmark

Students who are directly modeling the problem with tallies, marks, or objects and counting these objects do understand something about division but do not meet the benchmark at this point in fourth grade. Ask these students questions to help them see the groups of 8 in the representations they made and to connect these groups to known combinations. These students may need to do some or all of the following:

- Work on their multiplication combinations
- Work on division problems with smaller numbers to develop the idea of removing groups of the divisor
- Use story contexts to talk through the concept of division as removing groups of groups

Multiplying by Multiples of 10

One of the traditional mathematical "tricks" that help with hard multiplication was this "rule": When you multiply by 10, just add a 0; when you multiply by 100, add two 0s; and so on. Many students notice that when they multiply a multiple of 10 such as 6×40, they can "ignore" the 0, multiply the other digits (in this case, $6 \times 4 = 24$), and then "add a 0" to get an answer to the original problem. In this unit, students consider what "adding a 0" means and why it works. Although this pattern is a very useful one, it is critical that students grasp the underlying mathematical relationships in order to develop a solid understanding of multiplying by multiples of 10.

Our number system is based on powers of 10: ones, tens, hundreds, thousands, and so forth. The value of a digit in each place in a number in this system is 10 times greater than the value of the same digit to its right.

$$8 \times 1 = 8 \text{ ones} = 8$$
$$8 \times 10 = 8 \text{ tens} = 80$$
$$8 \times 100 = 8 \text{ hundreds} = 800$$
$$8 \times 1{,}000 = 8 \text{ thousands} = 8{,}000$$

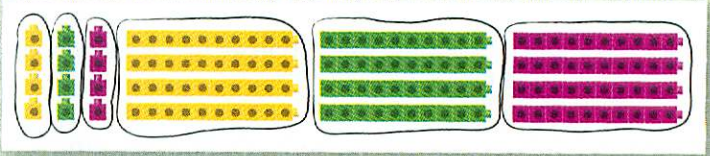
As the numbers in the last column increase, students see the addition of a 0 each time. In fact, what is happening is multiplication, not addition: Each number is 10 times greater than the previous number.

This progression becomes more difficult to see when multiplying multiples of 10, for example, 3×40. One way for you to think about how the product of 3×4 is related to the product of 3×40 is to look at the following series of equations:

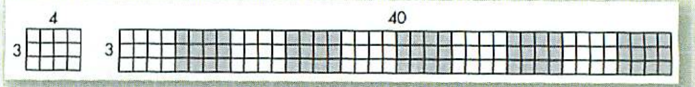
$$3 \times 4 = 12$$
$$(3 \times 4) \times 10 = 12 \times 10$$
$$(3 \times 4) \times 10 = 120$$
$$3 \times (4 \times 10) = 120$$
$$3 \times 40 = 120$$

This series demonstrates, through use of the associative property, that multiplying (3×4) by 10 is equivalent to multiplying 3×40. Therefore, the product of 3×40 is 10 times the product of 3×4.

Students are not likely to understand the relationship between 3×4 and 3×40 in this way, although some of them may be able to articulate that 3×40 is 10 times 3×4 because 40 is 10 times 4. For students, using arrays and other representations to show this relationship provides the best opportunity for them to see how multiplying by 10 and by multiples of 10 works. Consider the following two representations:



In this first representation, there are 12 cubes arranged in 3 groups of 4 cubes each: $3 \times 4 = 12$. Next to it are 3 groups with 4 towers of cubes in each group. There are 10 cubes in each tower. So there are 3×4 or 12 towers of cubes. Because each tower has 10 cubes, the total number of cubes is 12×10, or 120. In the first picture, there are 12 ones, or 12; in the second, there are 12 tens, or 120.



The second representation uses arrays to represent this relationship. The 3×4 array shows 3 rows of 4 squares. Ten of these smaller arrays are put together to create the 3×40 array. Now there are 3 rows of (4×10) squares. The 3×40 array has an area 10 times as large as the 3×4 array.

By visualizing and discussing representations such as these, students can develop a sound basis for understanding the "add a 0" rule.

Teacher Note

Reasoning and Proof in Mathematics

As students find strategies to perform calculations, they frequently make claims about numerical relationships. Part of the work of fourth grade involves helping students strengthen their ability to verbalize those claims and to consider questions such as these: Does this claim hold for *all* numbers? How can we know? Finding ways to answer these questions provides the basis for making sense of formal proof when it is introduced years from now. Consider the following vignette, in which a fourth-grade class is discussing methods for solving multiplication problems.

Andrew: When I did 12×25, I cut the 12 in half and doubled the 25 to make it 6×50. I can do 6×50 in my head. It's 300.

Teacher: Did anyone else use a strategy like Andrew's on any of the problems?

Lucy: I did. I was working on 14×15. I did the same kind of thing. I changed it to 7×30 and that's 210.

Teacher: Let's look at this. Are you saying that $12 \times 25 = 6 \times 50$? And that $14 \times 15 = 7 \times 30$?

Sabrina: The product stays the same. You cut one number in half and you double the other, so the answer is the same.

Teacher: Are you saying that this *always* works—that when you multiply two numbers, you can cut one number in half and double the other number, and the product will stay the same? Does the product stay the same no matter what the numbers?

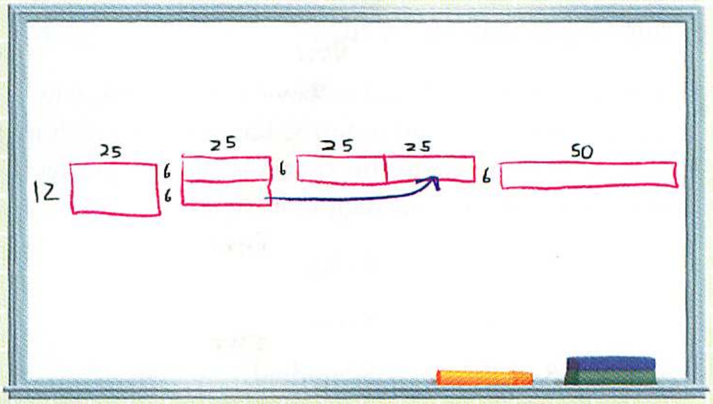
Sabrina: I think so.

Teacher: Can we find a way to use diagrams or cubes to show what is happening and why the product stays the same? You may begin with the examples we have seen, but you also must show how your argument will work for *all* numbers.

In this class, Sabrina made an assertion—mathematicians call such an assertion a *conjecture*—that the product of two numbers remains the same if you divide one of the numbers by 2 and double the other number. The teacher has challenged the class to find a way to show that this conjecture is true—not just for the examples they have noted, but for *all* pairs of numbers. If they can find a proof, they have what mathematicians call a *theorem* or *proposition.*

Let us return to the Grade 4 classroom to see how the students responded to their teacher's challenge to justify their conjecture.

Noemi: I made a diagram to show it.



Noemi: For Andrew's example, the first rectangle is 12 by 25. Then he cut the 12 in half and moved the bottom rectangle to make a rectangle that is 6 by 50. The first rectangle has the same area as the last one because they're both made up of the same two smaller rectangles. So like Andrew says, $12 \times 25 = 6 \times 50$.

Lucy: Noemi's picture works for my example too. You can think of the first rectangle as 14×15 and the last rectangle as 7×30.

Noemi: It doesn't matter what the numbers are. If you cut one in half, you would always have a rectangle on the bottom to move, and when you move that rectangle, you make a rectangle that is twice as long. It works for all numbers.

Noemi has presented a model of multiplication to show how she knows that $12 \times 25 = 6 \times 50$. Lucy sees that the same representation can be used to show $14 \times 15 = 7 \times 30$. Noemi then points out that, in fact, it doesn't matter what the numbers are. The model applies to *any* two numbers that are multiplied (provided that they are positive).

Note: The fact that Sabrina's conjecture has been shown to be true—if you double one factor and halve the other, you maintain the same product—does not necessarily mean that this doubling/halving strategy makes all multiplication easier. If both numbers are odd (e.g., 17×13), halving and doubling will result in one number that includes a fractional part (e.g., $34 \times 6\frac{1}{2}$). This is not a strategy one would choose to solve all multiplication problems. However, if both numbers are even, halving and doubling to create an equivalent problem can often lead to a simpler computation. Just as important, exploring *why* doubling and halving (and tripling and thirding, etc.) work provides an opportunity to learn more about the properties of multiplication and about developing mathematical justification.

Students in Grades K–5 can work productively on developing justifications for mathematical ideas, as Noemi does here. But what is necessary to justify an idea in mathematics? First we'll examine what "proof" is in the field of mathematics and then return to what kind of justification students can do in fourth grade.

What Is "Proof" in Mathematics?

Throughout life, when people make a claim or assertion, they are often required to justify the claim, to persuade others. A prosecutor who claims that a person is quilty must make an argument, based on evidence, to convince the jury of this claim. A scientist who asserts that the earth's atmosphere is becoming warmer must marshal evidence, usually in the form of data and accepted theories and models, to justify the claim. Every field, including the law, science, and mathematics, has its own accepted standards and rules for how a claim must be justified to persuade others.

When students in Grades K–5 are asked to give reasons why their mathematical claims are true, they often say things like this: "It worked for all the numbers we could think of." "I kept on trying and it kept on working." "We asked the sixth graders and they said it was true." "We asked our parents." These are appeals to particular instances and to authority. In any field, there are appropriate times to turn to authority (a teacher or a book, for example) for help with new knowledge or with an idea that we don't yet have enough experience to think through for ourselves. Similarly, particular examples can be very helpful in understanding some phenomenon. However, neither an authoritative statement nor a set of examples is sufficient to prove a mathematical assertion about an infinite class (say, all whole numbers).

In mathematics, a *theorem* must start with a mathematical assertion, which has explicit hypotheses ("givens") and an explicit conclusion. The proof of the theorem must show how the conclusion follows logically from the hypotheses. For instance, the fourth graders asserted that the product of two numbers remains the same if you divide one of the numbers by 2 and double the other number. In later years, their theorem might be stated as: If m and n are numbers, $m \times n = (\frac{m}{2}) \times (n \times 2)$. The proof of this claim consists of a series of steps in which one begins with the hypothesis—m and n are numbers—and follows a chain of logical deductions ending with the conclusion—$m \times n = (\frac{m}{2}) \times (n \times 2)$. Each deduction must be justified by an accepted definition, fact, or principle, such as the commutative or associative property of multiplication.

For example, to show that $m \times n = (m \times \frac{1}{2}) \times (n \times 2)$, we can develop this set of steps:

$$
\begin{aligned}
m \times n &= [m \times (\tfrac{1}{2} \times 2] \times n \\
&= [(m \times \tfrac{1}{2}) \times 2] \times n \\
&= (m \times \tfrac{1}{2}) \times (2 \times n)
\end{aligned}
$$

In this series of steps, the associative property of multiplication is applied twice. The associative property can be written with symbolic notation as $(a \times b) \times c = a \times (b \times c)$; regrouping the factors does not affect the product. For example, in the series of steps above, $m \times (\frac{1}{2} \times 2)$ can be regrouped as $(m \times \frac{1}{2}) \times 2$. It may help to look at how this works with one of the examples from the classroom dialogue:

$$\begin{aligned} 12 \times 25 &= [12 \times (\tfrac{1}{2} \times 2)] \times 25 \\ &= [(12 \times \tfrac{1}{2}) \times 2] \times 25 \\ &= (12 \times \tfrac{1}{2}) \times (2 \times 25) = 6 \times 50 \end{aligned}$$

The model for such a notion of proof was first established by Euclid, who codified what was known of ancient Greek geometry in his *Elements,* written about 300 B.C. In his book, Euclid begins with the basic terms and postulates of geometry and, through hundreds of propositions and proofs, moves to beautiful and surprising theorems about geometric figures. What is remarkable is that, in each mathematical realm, you can get so far with such simple building blocks.

What Does Proof Look Like in Fourth Grade?

One does not expect the rigor or sophistication of a formal proof, or the use of algebraic symbolism, from young children. Even for a mathematician, precise validation is often developed *after* new mathematical ideas have been explored and are solidly understood. When mathematical ideas are evolving and there is a need to communicate the sense of *why* a claim is true, then informal means of justification are appropriate. Such a justification can include the use of visual displays, concrete materials, or words. The test of the effectiveness of such a justification is this: Does it rely on logical thinking about the mathematical relationships rather than on the fact that one or a few specific examples work?

This informal approach to mathematical justification is particularly appropriate in Grade K–5 classrooms, where mathematical ideas are generally "under construction" and where sense-making and diverse modes of reasoning are valued. Noemi's argument offers justification for the claim that if you halve one factor and double the other, the product remains the same. The product of the numbers m and n is represented by the area of a rectangle with dimensions m and n. Noemi then cuts the vertical dimension in half, making two rectangles, each having dimensions $(\frac{m}{2})$ and n. One of these rectangles is moved and then connected with the other to create a rectangle with dimensions $(\frac{m}{2})$ by $2 \times n$. The area of this new rectangle must be the same as the original, therefore $(\frac{m}{2}) \times (2 \times n) = m \times n$. Noemi's argument establishes the validity of the claim not only for particular numbers, but for any numbers, and easily conveys why it is true.

An important part of Noemi's justification is her statement that it does not matter what the numbers are. She understands that the process she describes with her model will guarantee that the original rectangle will have the same area as the final rectangle whose length is double that of the original and whose width is half that of the original. It is important to note that when students make such claims of generality—*this is true for all numbers*—the phrase *all numbers* refers to the numbers they are using. In this vignette, Noemi's reasoning about multiplication takes place in the context of whole numbers. We might see that Noemi's argument applies equally well to positive values that include rational numbers, but Noemi and her classmates will need to revisit this argument when the domain of numbers they are working with expands beyond whole numbers.

To support the kind of reasoning illustrated in the vignette, encourage students to use cubes, number lines, and other representations to explain their thinking. The use of representations offers a reference for the student who is explaining his or her reasoning, and it also allows more classmates to follow that reasoning. If it seems that students may be thinking only in terms of specific numbers, you might ask such questions as these: Will that work for other numbers? How do you know? Will the explanation be the same?

Teacher Note

Multiplication Clusters and the Properties of Multiplication

Multiplication clusters are sets of problems that help students think about using what they know to solve harder problems. For example, what do you know that helps you solve 12×3? If you know that $3 \times 3 = 9$, you can double the product of 3×3 to get the product of 6×3 and then double again to get the product of 12×3. You might also start with 10×3. If you know that $10 \times 3 = 30$, then you can start with 30 and add two more 3s to get 36. As students work with multiplication clusters, they learn to think about all the number relationships they know that might help them solve a problem.

The multiplication clusters in this unit are designed to help students make sense of multiplying 2-digit numbers. Many of the clusters build an understanding of pulling apart multiplication problems into manageable subproblems, solving the smaller problems, and then putting the parts back together. This process is based on an important characteristic of multiplication called the *distributive property.* In this unit, students are not expected to learn the name of the property, but it is a core idea of the unit.

Here is an example:

$$6 \times 23 = (6 \times 10) + (6 \times 10) + (6 \times 3)$$

In this example, 23 is broken apart into $10 + 10 + 3$, and *each part* is multiplied by 6 in order to construct the solution to 6×23. The number does not have to be split into 10s and 1s.

Here is another example:

$$8 \times 12 = (4 \times 12) + (4 \times 12)$$

or

$$8 \times 12 = (8 \times 6) + (8 \times 6)$$

In each case, one of the factors is split up into parts, and each part is multiplied by the other factor in order to maintain equivalence to the original expression.

Other clusters build on ideas about halving and doubling that are developed in this unit. See **Teacher Note:** Reasoning and Proof in Mathematics?, page 168, for more about students' understanding of creating an equivalent multiplication problem by halving one factor and doubling the other.

As students solve the first few problems in each cluster, they use familiar multiplication combinations. Students say "I just knew it" for some of the problems because these single-digit multiplication combinations are part of their known repertoire. They also make use of multiplying by 10 and by multiples of 10, another essential tool in solving harder multiplication problems. See **Teacher Note:** Multiplying by Multiples of 10, page 167, for more about the ways students develop understanding of this idea.

Here are examples of student work on two multiplication clusters from the *Student Activity Book* pages 57–58.

Set C Solve these problems. How did you solve the final problem?

32 × 2 = 64

10 × 8 = 80

30 × 8 = 240

Final problem: **32 × 8** = 256

I broke the 32 into 30 + 2. I already solved 30 × 8 = 240 + I need 2 more groups of 8 which is 16.

240 + 16 = 256

Sample Student Work

Set D Solve these problems. How did you solve the final problem?

63 × 10 = 630

60 × 11 = 660

3 × 11 = 33

Final problem: **63 × 11** = 693

I knew 63 × 10 = 630 + I need to add one more group of 63 which makes 693.

Sample Student Work

Multiplication clusters help students learn how to look at a problem and build a strategy to solve it that is based on the number relationships they know. At first, students work on clusters of problems that are provided to help them solve a 2-digit problem, such as 4 × 43 or 58 × 6. They solve all the problems in the cluster and then decide which one(s) will most help them think about the solution to the final problem. Students may add to the cluster any other problems that help them solve the final one. Later in the unit, students create their own cluster of problems to help them solve a multiplication problem. In later units of *Investigations,* in both Grades 4 and 5, students spend more time creating their own clusters of problems as well as using a variety of given problems to solve multiplication and division problems.

End-of-Unit Assessment

This final assessment focuses on four of the five benchmarks for the unit. (The fifth benchmark, "Know multiplication combinations to 12 × 12 fluently," was the focus of the assessment activity Multiplication Combinations in Session 3.4.)

At this point in Grade 4, students' work on multiplication is focused on solving problems by breaking apart the numbers to create multiplication problems that are easier to solve and then recombining the products of those problems. The focus for division is on understanding that groups of the divisor can be removed in groups from the number being divided.

Problem 1

Benchmark addressed:

Benchmark 1: Multiply 2-digit numbers by 1-digit and small 2-digit numbers (e.g., 12, 15, 20), using strategies that involve breaking the numbers apart.

In order to meet the benchmark, students' work should show that they can:

- Interpret the problem as 6 groups of 23 or 23 groups of 6;
- Accurately solve the problem by using multiplication (e.g., students should not be adding or counting groups).

Name Date

Multiple Towers and Division Stories

End-of-Unit Assessment (page 1 of 2)

Problem 1

Solve this problem and show your solution clearly:

23 × 6

▲ Resource Masters, M51

Meeting the Benchmark

Students who meet the benchmark can break the problem into smaller problems that they know how to solve. They solve each part correctly and accurately combine all the parts for the solution.

Ramona breaks 23 into tens and ones, 20 + 3, and multiplies each part by 6. She then combines the two parts to solve the problem (20 × 6 = 120, 3 × 6 = 18, 120 + 18 = 138). Many fourth graders should be able to complete these calculations mentally, simply recording each part to keep track.

(20×6)+(3×6)=23×6
120 + 18 = 138

Ramona's Work

Some students break the 6 into 3 + 3. Luke, for example, multiplied 23 × 3 = 69. He knew that 23 × 6 = (23 × 3) + (23 × 3), so he added 69 + 69 to get his final answer.

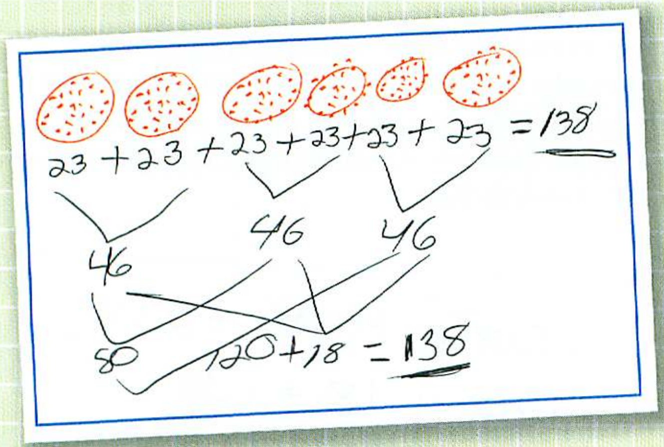
I know that 23×3=69
so I just added 69+69
which =138.

Luke's Work

Partially Meeting the Benchmark

Some students still use addition to solve a multiplication problem (e.g., adding up groups of 23), as they may have done when multiplying in Grade 3. These students understand what the multiplication expression 23 × 6 represents and have a strategy for finding the correct answer.

Alejandro drew 6 circles of 23 and then added 23 six times.



Alejandro's Work

Work with these students to identify multiplication combinations they know that are related to the problem they are trying to solve. In this example, does Alejandro know that 3 × 20 = 60 or that 6 × 20 = 120? Some fourth graders continue to add because they are more secure and confident with addition. However, they do know multiplication combinations that can help them solve a problem such as this one and need to be supported to do so. Other students need more work on their basic multiplication combinations and on multiplying multiples of 10.

Not Meeting the Benchmark

Some students may not understand the meaning of multiplication as equal groups (e.g., solving 23 + 6 instead of 23 × 6). These students should solve and create story problems to clarify what a multiplication expression such as 23 × 6 represents: How many yogurts are in 23 six-packs?

Other students may understand the problem but cannot break apart the problem in any useful way. They may set up 23 groups of 6 cubes and count them 1 by 1. These students should be encouraged to create any groups that make sense to them. Ask them questions such as these:

"Do you know what 2 groups of 6 are? 10? 20? Can you find groups of 23 instead? How many do you need?"

These students may need more work on their basic multiplication combinations and on *multiplying multiples of 10*.

Problem 2

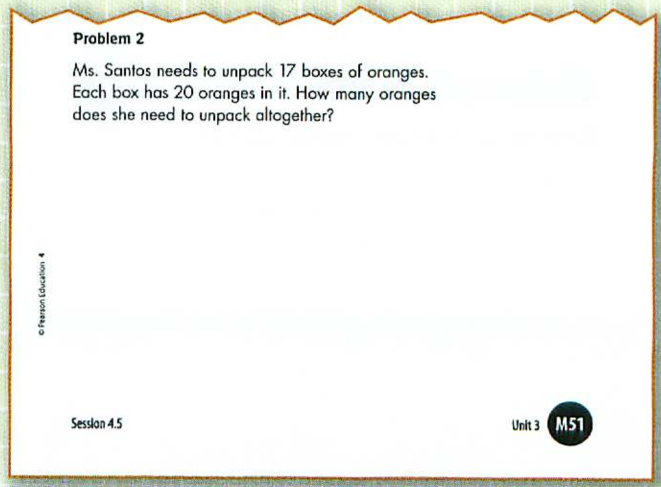
Benchmarks addressed:

Benchmark 1: Multiply 2-digit numbers by 1-digit and small 2-digit numbers (e.g., 12, 15, 20), using strategies that involve breaking the numbers apart.

Benchmark 4: Multiply by 10 and multiples of 10.

In order to meet the benchmarks, students' work should show that they can:

- Interpret the problem as a multiplication situation;
- Break the problem apart in a useful and efficient way, solve the parts, and combine them to find the correct answer;
- Multiply by multiples of 10.



Problem 2

Ms. Santos needs to unpack 17 boxes of oranges. Each box has 20 oranges in it. How many oranges does she need to unpack altogether?

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Session 4.5 Unit 3 M51

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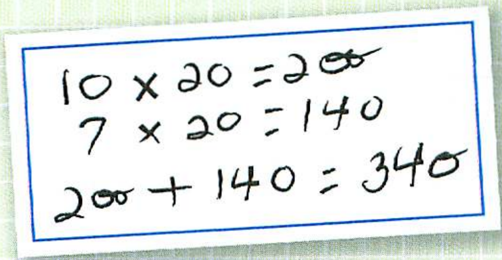
Meeting the Benchmarks

Students who meet the benchmarks interpret this problem as multiplication and have efficient ways to break the numbers in the problem apart, using multiplication by 10 or 20.

Richard started with $17 \times 10 = 170$ and then doubled 170 to get the correct answer of 340.

Jill knew that 17×20 can also be written as $17 \times 2 \times 10$. She multiplied $17 \times 2 = 34$ and then $34 \times 10 = 340$.

Derek broke 17 into 10 and 7 and multiplied each piece by 20.



Derek's Work

Some students might use doubling or halving to create the equivalent problem 34×10, which they then solve directly, applying what they know about multiplying by 10.

Partially Meeting the Benchmarks

Some students interpret the problem correctly as 17×20 and understand how to break apart the numbers to make them easier to solve, but make a computation error (e.g., add $170 + 170$ incorrectly).

Not Meeting the Benchmarks

Multiplying by 10 and multiples of 10 is a necessary tool for efficiently solving multiplication and division problems. Some students do not break apart the problem in useful ways or use knowledge of multiplication by 10s. They may multiply 2 groups (or some other amount) of 17 and add these until they have 20 groups of 17. Alternatively, they may solve 17×10 by adding up groups of 17 instead of readily knowing that product.

Problem 3

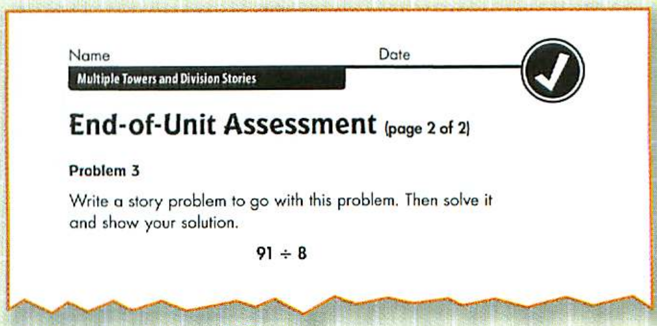
Benchmarks addressed:

Benchmark 2: Solve division problems (2-digit and small 3-digit numbers divided by 1-digit numbers), including some that result in a remainder.

Benchmark 3: Use story contexts, pictures, or concrete models to represent division situations.

In order to meet the benchmarks, students' work should show that they can:

- Interpret division notation and understand how it relates to a context;
- Solve a division problem accurately by using groups of the divisor or splitting the dividend into parts;
- Make sense of the effect of the remainder on the solution to the story problem.


Name Date
Multiple Towers and Division Stories

End-of-Unit Assessment (page 2 of 2)

Problem 3

Write a story problem to go with this problem. Then solve it and show your solution.

$91 \div 8$

▲ Resource Masters, M52

Meeting the Benchmarks

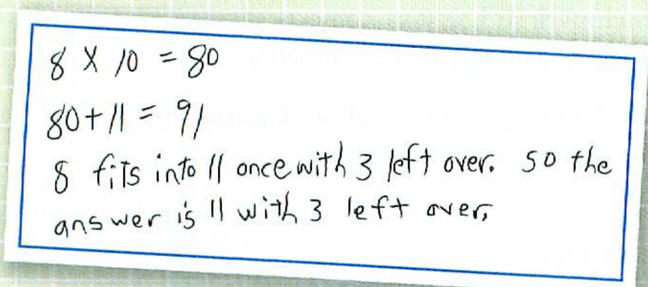
All students should be able to write a story problem in which 91 of something is split into either 8 groups or groups of 8. Here are two examples of story problems that can be represented by 91 ÷ 8:

I have 91 muffins and 8 friends. How many muffins will each friend get?

There are 91 people who need to fit into cars that hold 8 passengers. How many cars do we need?

The following students all used groups of the divisor to solve a story problem similar to the muffin problem above. They used multiplication combinations they knew to help them find how many groups of 8 there are in 91.

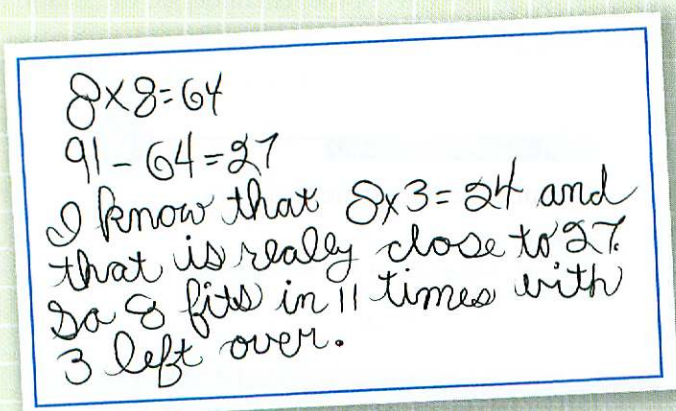
Andrew first multiplied 8 × 10.



8 X 10 = 80
80+11 = 91
8 fits into 11 once with 3 left over. So the answer is 11 with 3 left over.

Andrew's Work

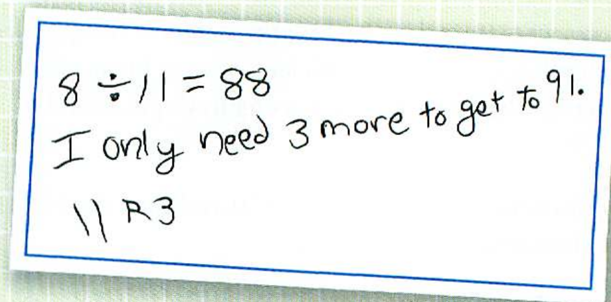
Helena used 8 × 8 = 64 as a first step.



8x8=64
91-64=27
I know that 8x3=24 and that is realy close to 27. So 8 fits in 11 times with 3 left over.

Helena's Work

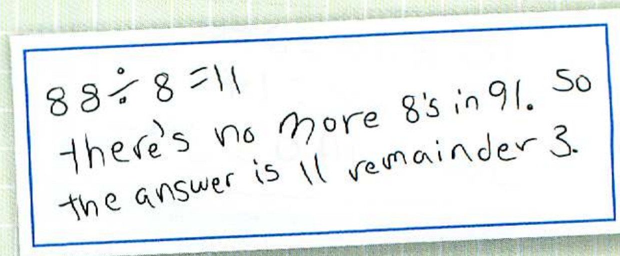
Alejandro uses yet another familiar fact, 8 × 11.



8 ÷ 11 = 88
I only need 3 more to get to 91.
11 R3

Alejandro's Work

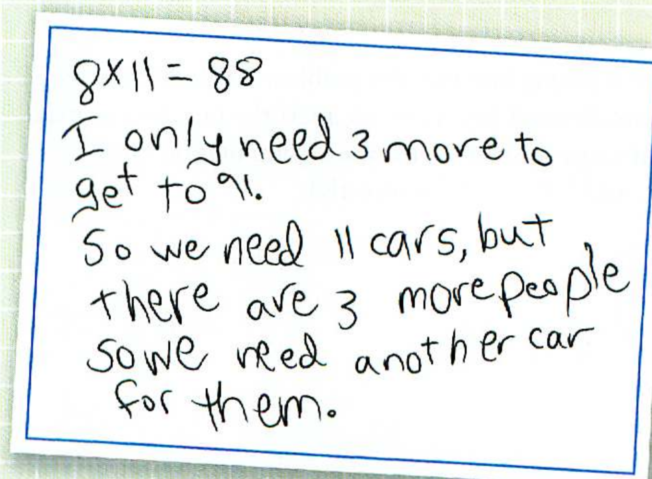
Some students divide, starting with 80 ÷ 8, or they know that 88 ÷ 8 is the closest division by 8 that results in a whole number.



88 ÷ 8 = 11
there's no more 8's in 91. So the answer is 11 remainder 3.

Steve's Work

If students choose a story problem similar to the problem about cars above, they should be able to take into account how the remainder affects the answer to the problem. So, for example, if cars are involved, they might write:

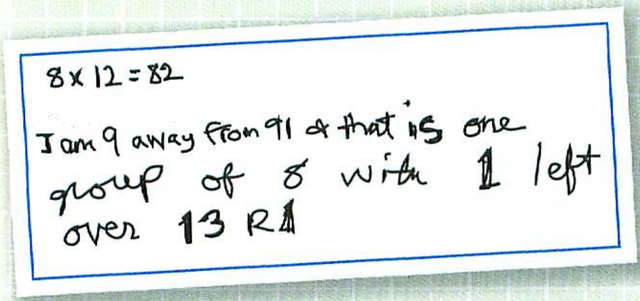


8x11=88
I only need 3 more to get to 91.
So we need 11 cars, but there are 3 more people so we need another car for them.

Kimberly's Work

Partially Meeting the Benchmarks

Some students understand that they can use multiplication combinations they know but miscalculate them. For example, a student might solve 8 × 12 incorrectly, thus arriving at the wrong answer.



Marisol's Work

This student can interpret the division expression and can use multiplication to solve a division problem, but he needs to work on basic multiplication combinations and on double-checking his work.

Some students subtract 8s or groups of 8. For example, Noemi knows that she is trying to figure out how many 8s will fit into 91. She keeps subtracting 8 until she has no more to distribute, as follows:

91 − 8 = 83

83 − 8 = 75

75 − 8 = 67

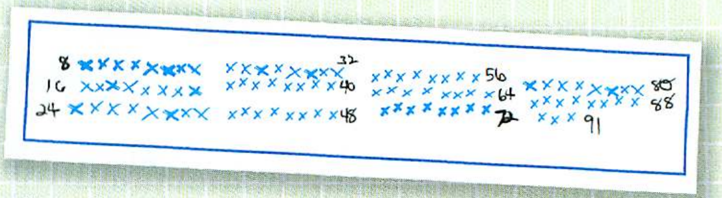
67 − 8 = 59 . . .

She continues all the way to 3 and then counts up the number of 8s she has subtracted, getting the correct answer, 11 R3. Noemi can interpret the division expression 91 ÷ 8. However, at this point in Grade 4, students should be able to solve a division problem with numbers of this magnitude by using a strategy based on larger groups of 8s.

Not Meeting the Benchmarks

Students who are directly modeling the problem with tallies, marks, or objects and counting these objects do understand something about division but do not meet the benchmark at this point in fourth grade.

Jill's work is an example of this strategy.



Jill's Work

This strategy may work to arrive at the correct answer but, it is inefficient and prone to error. In addition, use of this approach indicates that students do not have an understanding of how to use groups of 8 to help them solve the problem.

These students may need to do some or all of the following:

- Work on their multiplication combinations
- Work on division problems with smaller numbers to develop the idea of removing groups of the divisor
- Use story contexts to talk through division as removing groups of groups

Solving 17×6

Students have been working in pairs to think of ways that they can solve the problem 17×6 by breaking it into smaller problems. As the teacher watched them work, she noticed that they came up with a number of different ways to break up the problem. She lists some of these on the board and then brings the students together for a discussion.

$$(17 \times 2) + (17 \times 2) + (17 \times 2)$$

$$(6 \times 11) + (6 \times 3) + (6 \times 3)$$

$$(17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1)$$

$$(6 \times 10) + (6 \times 7)$$

Teacher: Here are some of the ways I noticed you using to break 17×6 into easier problems. All of these are correct, and I'm wondering how you chose ways to break up 17×6 that helped you solve the problem. Were you thinking of smaller problems that you already knew to help you solve 17×6? What did you do to make this problem easier?

Richard: I started with 17×2 because I knew that the answer is 34. Then all I had to do was add $34 + 34 + 34$.

Venetta: I think $6 \times 11 = 66$ is easy, and then all I had to do was add 6 more 6s. I did 3 groups of 6 and then 3 more groups of 6.

Teacher: Many of you know the multiples of 11 and 3×6. Who can tell me another way that made it easy for you?

Steve: 17×1.

Teacher: It is easy. What do you have to do next to finish solving the problem?

Steve: You still have to add six 17s together. That might take a long time. But it would work!

Teacher: That's right, it would work, but you're pointing out that it leaves you with some fairly difficult addition. One question everyone can think about when you're breaking problems apart is this: What do you know that can help you solve a pretty big chunk of the problem? Here's another question: Do you need to break the problem apart into many pieces, or can you find an efficient way to use only two or three pieces? When you look at all the ways that are listed here, which do you think are both easy and efficient ways to solve the problem?

Cheyenne: I think 6×10 plus 6×7 is pretty efficient because I know most of the 10 times tables. The 7s are a little harder, but 6×7 is easy for me.

Teacher: Exactly. You all know multiples of 10, and you can do 6×7 in your head. You can be more efficient when you use what you already know.

By asking students to compare different ways they broke problems apart, this teacher helps students consider how to use what they know to solve more difficult problems easily and efficiently. These students understand how to break a multiplication problem into smaller problems. Some students are also using their knowledge of the commutativity of multiplication—they find it easier to think of 17×6 as 6×17. It is important that students understand that there are many ways to break up a multiplication problem. As the teacher points out, all of their methods are correct. However, they can also think about which ways of breaking up the problem are most helpful.

Venetta knows how to solve 6×11; she also knows that she needs "6 more 6s," which she thinks of as 3 groups of 6 and another 3 groups of 6. However, breaking 6×17 into these three parts leaves her with the addition problem, $66 + 18 + 18$, which might not be easy to solve.

Steve also notices that he has not made the problem easier to solve when he breaks it up into $(17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1) + (17 \times 1)$.

Cheyenne breaks the problem into two parts that are each easy for her to solve. She has to add only two numbers to find the solution to the original problem. Another student, who knows all the 12s multiplication combinations well, might break the problem into $(12 \times 6) + (5 \times 6)$, which also results in a relatively easy addition to complete the problem.

As students continue to solve problems with larger numbers by breaking them into smaller problems, this teacher encourages them to use problems that they can solve easily, such as multiplication combinations they know and multiplying by 10. She will continue to ask them to compare solution methods and to think about which ways are easy and efficient to carry out.

Dialogue Box

What Do You Do with the Extras?

The students in this classroom are working on *Student Activity Book* pages 21–22. The teacher listens to students as they work on Problem 1.

There are 44 people taking a trip in some small vans. Each van holds 8 people. How many vans will they need?

Lucy: It can't be 6 vans. No, there'd be 48 people. 2 vans are 16 people, 16 + 16 is 32. 32 plus 8 is 40. There will be 5 vans. That's weird, though. 5 vans with 8 people and 1 van with 4 people.

Lucy writes on her paper:

5 vans will have 8 people.
1 van will have 4 people.

Teacher: How did you decide on your answer?

Lucy: 5 × 8 is 40, and that's the closest thing under 44 that's a multiple of 8. So 5 vans will have 8 people, and 1 van will have 4.

Teacher: What would the division equation for that be?

Lucy: 44 ÷ 8. The answer would be 5 with 1 left over. No, 5 with 4 left over.

Teacher: 4 what?

Lucy: 5 × 8 is 40. Then there are 4 people left. There are 5 vans but 4 extra people.

As the teacher observes the students working, she recognizes that they are having difficulty sorting out the difference between the answer to the question posed in Problem 1 and the way to complete the division equation 44 ÷ 8 = _____. She decides to stop the class to discuss this issue.

Derek: There are two ways you could deal with this problem. You could just take 5 vans and leave the 4 people behind or you could take 1 extra van using only 4 spaces.

The students agree that it would not be fair to leave 4 people behind.

Teacher: Derek said that you could have 5 full vans with 8 people and 1 van with the 4 extra people. So what would be the answer to the question "How many vans will they need?"

Amelia: They would need 6 vans if they want to take all the people, but 6 isn't the answer to 44 ÷ 8.

Teacher: What do you mean, that 6 is not the answer to 44 ÷ 8?

Lucy: It can't be, because 6 is the answer to 48 ÷ 8.

Ramona: I got 5 with a remainder of 4.

Abdul: You can also say $5\frac{1}{2}$, like with the cracker problem.

Teacher: Are you saying that these equations are correct?

The teacher records:

$44 \div 8 = 5\ R4 \qquad 44 \div 8 = 5\frac{1}{2}$

Ramona, Abdul, and others: Yes.

Teacher: So $5\frac{1}{2}$, or 5 remainder 4, is the answer to 44 ÷ 8, but is it the answer to how many vans they need?

Abdul: No. You can't drive a half van and it's not fair to leave 4 people behind, so they need 6 vans.

Teacher: Sometimes in problems about real situations, the answer to the question being asked is not the same as the number you would write to complete the division equation. I have one more question for you: If the answer to the problem is 6 vans, can I write this?

The teacher records: 44 ÷ 8 = 6

Various students: No. Yes. No.

Teacher: Lucy said something before about 48 ÷ 8 = 6.

Enrique: Right, you can't say 44 ÷ 8 = 6 because 48 ÷ 8 = 6. They can't both be 6.

Marisol: They wouldn't be equal. It's 6 vans, but it's not really 6 because one of the vans has only half the people.

In this discussion, the teacher helps students sort out how the answer to the problem about vans is not the same as the number needed to complete the equation 44 ÷ 8 = ____. By considering different division situations, students gain experience in attending carefully to the question posed in the problem. When a problem involves whole objects that cannot be broken into parts, the answer to the question in the problem cannot include a fraction or decimal. Sometimes, as in the van problem, the result of the division must be rounded up to the next whole number: 6 vans are needed. Sometimes, as in a situation involving sharing 44 balloons among 8 people, the result of the division must be rounded down: each person cannot have more than 5 balloons, if each person is to have an equal number of balloons.

The teacher's primary focus in this discussion is on finding the solution to a division problem posed in a context. The teacher also noticed that quite a few students had written the incorrect equation, 44 ÷ 8 = 6, on their papers. She raises the question about whether this equation is correct at the end of the discussion, hoping that some students will be able to build on Lucy's observation that 48 ÷ 8 = 6. She knows that she will need to return to this idea.

Dialogue Box

Building a Multiple Tower

The students in this class are building a tower of paper squares as tall as their teacher, using the multiples of 30. As students call out the multiples, one student lists them on the board and another writes them on self-stick notes and puts them on the wall next to their teacher. The tower is about knee-high.

Teacher: Right now we are at 240 and the tower is up to my knee. What number do you estimate we'll land on when the tower is as tall as I am? Write your estimate on a slip of paper.

Sabrina: I think maybe 554.

Teacher: Read out the multiples we have so far.

Bill: 30, 60, 90, 120, 150, 180, 210, 240.

Teacher: Does 554 fit the pattern?

Jake: No, because all the numbers end in a 0. So I don't think it could be 554.

Jill: But it could be 550 or 680.

Luke: If you look at these numbers (referring to the tens place), you add 3 each time.

Teacher: Let's think about which estimates are possibilities and which don't fit the pattern.

The students recheck their estimates, using the pattern as a guide. Some decide to alter their estimates, and the tower building continues until it reaches the teacher's height at 960.

Teacher: So we skip counted by 30s and I am 960. Can you figure out how many multiples are in my tower?

The teacher writes on the board: ____ × 30 = 960

Amelia: Let's see. It's ten 30s to 300, so 20 to 600 and 30 to 900. Plus it's 2 more 30s to 60. So it's thirty-two 30s to 960.

Richard: I did it in the calculator and got 32, too.

Marisol: I broke it into 900 and then 60. Then I said, "How many times does 3 go into 9?" I know it's 3, so it's 30 into 900. Then it's 2 more to get to 960. So it's 32.

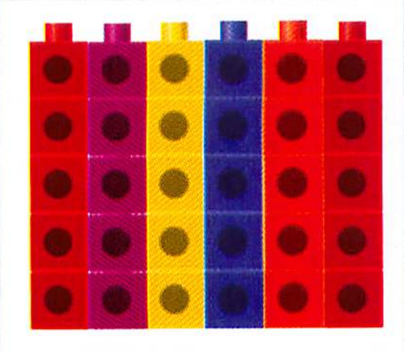
In this discussion, students use skip-counting patterns to help them make reasonable predictions about multiples of 30. They also consider what they already know about 30s—that there are 3 in 90 or 10 in 300—to help them solve the problem "How many 30s in 960?" Just as students can break multiplication problems into parts, they can also break division problems into parts that are easier to solve.

Students may not yet be thinking of their work on the multiple towers as division, but the question "How many 30s are in 960?" can be represented as 960 ÷ 30. As they break up the dividend (960) into parts and divide each part by 30, they are developing strategies that they can apply to division problems.

Dialogue Box

What Does It Mean to "Add a Zero"?

Students have completed *Student Activity Book* pages 42–43. They have gathered to share their representations for 5 × 6 and 5 × 60. The teacher holds up a 5 by 6 rectangular array that one student made with connecting cubes.



Teacher: How many more of these (the 5 by 6 array) do you need to make 5 × 60?

The class decides to skip count together by 30s.

Anna: We need 10 of those.

Teacher: It seems that many people say "add a zero" to explain what the relationship is. Can anyone explain what that really means?

Benson: You're adding a 10.

Derek: No, times 10.

Teacher: You decided that we need 10 of these 5 × 6 arrays to make a 5 × 60 array. Does this make sense then?

The teacher writes on the board: $(5 \times 6) \times 10 = 5 \times 60$

The students puzzle over it for a bit but then decide that both parts of the equation are equal to 300.

Steve: You know 5 × 6 = 30, so if you multiply 30 × 10 you get 300.

Teacher: Instead of 5 × 6 and 5 × 60, what if I have 6 × 5 and 6 × 50? You know that 6 × 5 = 30, so what should the answer be for 6 × 50?

Helena: I think the relationship is the same. This time you're making the 5 ten times bigger.

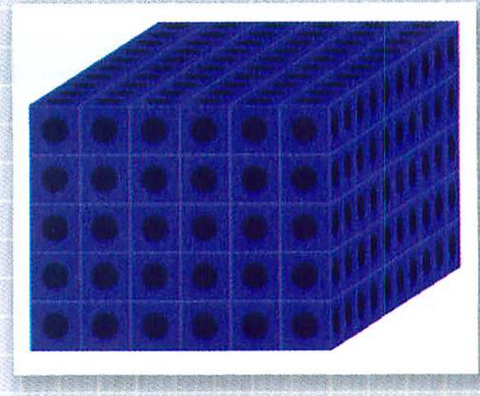
Andrew: They're in groups. First they're in groups of 5 and then they're in groups of 50.

Teacher: What do you mean?

Lucy: The zero really means 10. It's 5 × 10.

Teacher: Damian, you had a sketch of a story problem that might help us see what Andrew is saying about groups. Can you explain it?

The teacher continues to have students share representations, which include ten 5 × 6 arrays cut from graph paper and put together to make a 5 × 60 array and a cube construction with the dimensions of 5 × 6 × 10. Kimberly points to one of the 5 × 6 faces of her 3-D cube construction.



Kimberly: See, this is 5 × 6, but if you want to do 5 × 60, you can see that there's really 10 in back of every one of these, so it's ten 30s, which makes 300.

At this point, not all students can articulate the idea behind "adding a zero" as clearly as some of the students in this discussion, so the teacher decides to leave the representations on display. He will continue to ask students to explain what the zero represents as students use this pattern in their problem solving. See **Teacher Note:** Multiplying by Multiples of 10, page 167.

Student Math Handbook

The *Student Math Handbook* pages related to this unit are pictured on the following pages. Encourage students to use the **Math Words and Ideas** pages as a summary of the math content covered in class. Remind students to think about and answer the question(s) at the bottom of many of these pages. Students can use the **Games** pages to review game directions during class or at home.

When students take the *Student Math Handbook* home, they and their families can discuss these pages together to reinforce or enhance students' understanding of the mathematical concepts and games in this unit.

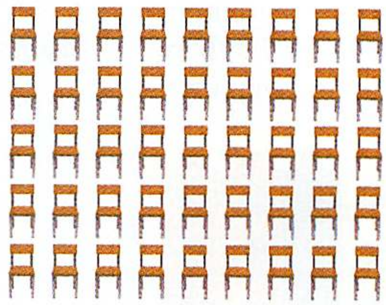
Math Words and Ideas

Arrays (page 1 of 2)

Math Words
- array
- dimension

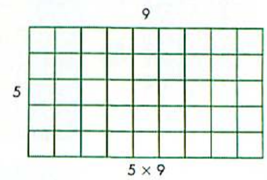
An array is one way to represent multiplication.

Here is an array of chairs.
There are 5 rows of chairs.
There are 9 chairs in each row.



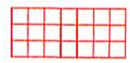
The arrangement of chairs can be represented as a rectangle.

When we talk about the size of an array, we say that the dimensions are "5 by 9" (or "9 by 5," depending on how you are looking at the array).



5 × 9

What are the dimensions of this array?

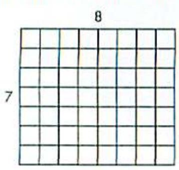


Math Words and Ideas, p. 18

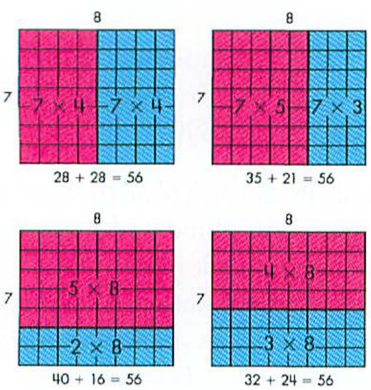
Math Words and Ideas

Arrays (page 2 of 2)

Here are some examples of rectangular arrays that show how multiplication problems can be broken into smaller parts.



This 7 by 8 array can be broken into parts to find the product in many different ways.



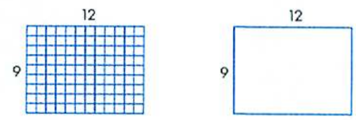
All of these arrays show that the product of 7 × 8 is 56.

Math Words and Ideas, p. 19

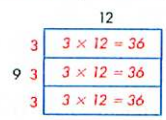
Math Words and Ideas

Unmarked Arrays (page 1 of 2)

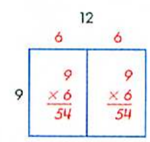
With larger numbers, unmarked arrays can be easier to use than arrays with grid lines. You can imagine the rows of squares without drawing them all.



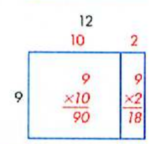
Look at the ways that unmarked arrays are used to show different ways to solve the problem 9 × 12.



9 × 12 = (3 × 12) + (3 × 12) + (3 × 12)
9 × 12 = 36 + 36 + 36
9 × 12 = **108**



9 × 12 = (9 × 6) + (9 × 6)
9 × 12 = 54 + 54
9 × 12 = **108**



9 × 12 = (9 × 10) + (9 × 2)
9 × 12 = 90 + 18
9 × 12 = **108**

Math Words and Ideas, p. 20