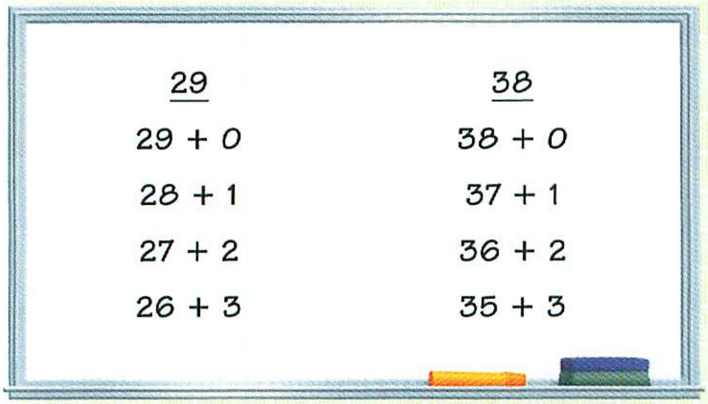
Teacher Note

Reasoning and Proof in Mathematics, Part 1

Second-grade students often find numerical relationships as they work with numbers and manipulatives. You should encourage them to do so and verbalize these patterns and discoveries. Consider such questions as these: Does this hold for *all* numbers? How can we know? Examining these questions will provide the basis for the formal mathematical idea of proof when it is introduced years from now.

Consider the following vignette in which students discuss a pattern that comes up in the *Today's Number* routine:

Teacher: Today's Number is 38. A few days ago it was 29. Here's what several of you wrote when 29 was Today's Number and what you have done today for 38.



Teacher: What do you notice? Is there a pattern?

Anita: You take 38, and then you are adding zero. And then you take one number less than the 38 and one number more than the zero.

Teacher: And what happens next?

Henry: You take 38 + 1, and then you take one less than 38 and one more than one. That's 37 + 2.

Rochelle: It's the same thing with 29. You take one off one number and give it to the other.

Teacher: Do all of these expressions on the right equal 38? And all of these on the left equal 29?

Henry: They are. You're just taking one from one number and giving it to the other.

Teacher: Does it work for only 38 and 29? Or are you saying that you can *always* do that?

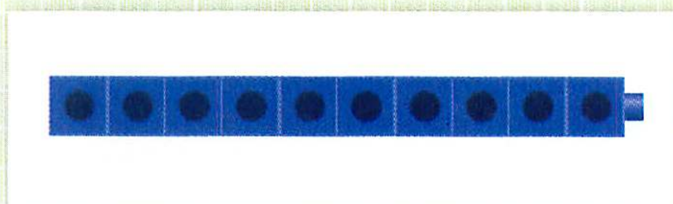
Rochelle: You can always do it.

Teacher: Show me, with a picture or with cubes, how you know that. When you're adding any two numbers, you can take one from one number and add it to the other and get the same total.

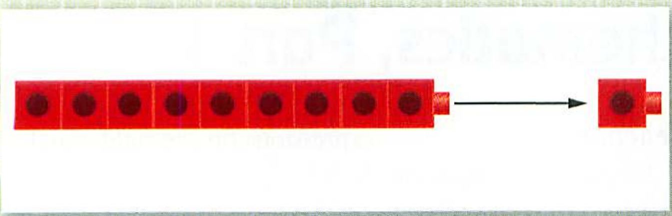
The students have made an assertion—mathematicians call such an assertion a conjecture—that the total of two addends remains the same if you subtract 1 from one addend and add 1 to the other. The teacher has challenged the class to show that this conjecture is true not just for 38 and 29 but for *all* numbers.

How do second graders respond to such a question? Consider how the students above responded to their teacher's challenge to justify their conjecture. Alberto begins, using a tower of ten cubes to demonstrate his ideas.

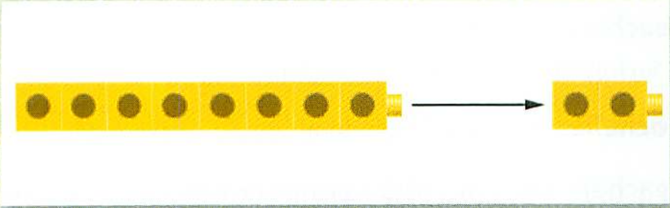
Alberto: I started with 10 + 0.



If you move one over it's 9 + 1.



Move another one over, and now it's 8 + 2.



Rochelle: You can keep going. You take 1 from one number and give it to the other.

Teacher: Alberto used ten cubes. What if he started with 38? Could he still subtract 1 from one number and add it to the other?

Anita: Yes. Because you're still using the same amount, the same number of cubes.

Henry: It doesn't matter how many cubes you start with. It will still work because you're just changing them around. You keep taking from one number and adding it to the other.

Tia: It has to be the same because, whatever number you start with, you don't put any more cubes in and you don't take any away.

Alberto presents a model of addition—joining two sets of cubes—that shows how he knows that $10 + 0 = 9 + 1 = 8 + 2$. Anita sees that the same representation can show why $38 + 0 = 37 + 1 = 36 + 2$. Henry and Tia point out that it does not matter what the original numbers are. The model could be used to make a similar argument for any two whole numbers. Later these students might refer back to such visual models to show why the same action—subtracting from one number and adding to the other—does not produce the same results with other operations.

K–5 students can develop justifications for mathematical ideas just like these Grade 2 students. But what is necessary to prove an idea in mathematics? What does proof look like in Grade 2?

What Is Proof in Mathematics?

Throughout life, when people make a claim or assertion, they often need to justify the claim to others. A prosecutor who claims that a person is guilty of a crime must make an argument, based on evidence, to convince a jury of this claim. A scientist who asserts that the earth's atmosphere is becoming warmer must marshal evidence in the form of data, accepted theories, and models to justify the claim. Every field, including the law, science, and mathematics, has its own accepted standards and rules for how a claim must be justified in order to persuade others.

When K–5 students are asked to give reasons why their mathematical claims are true, they often say such things as "It worked for all the numbers we could think of," "I kept on trying and it kept on working," "We asked the sixth graders and they said it was true," or "We asked our parents." These are appeals to particular instances and to authority. In any field, there are appropriate times to turn to authority (a teacher or a book, for example) for help with new knowledge or with an idea that we do not yet have enough experience to think through for ourselves. Similarly, particular examples can be very helpful in understanding some phenomena. However, neither an authoritative statement nor a set of examples is sufficient to prove a mathematical assertion about an infinite class (such as all whole numbers).

In mathematics, a *theorem* must start with a mathematical assertion, which has explicit hypotheses (or "givens") and an explicit conclusion. The proof of the theorem must show how the conclusion follows logically from the hypotheses. A mathematical argument is based on logic and gives a sense of why a proposition is true. For instance, the second graders asserted that the sum of two addends remains the same if you subtract 1 from one addend and add 1 to the other addend. In later years, their theorem may be stated as follows: If m and n are numbers, $m + n = (m - 1) + (n + 1)$. The proof consists of a series of steps starting with the hypothesis—m and n are numbers—and follows a chain of logical deductions ending with the conclusion—$m + n = (m - 1) + (n + 1)$. Each deduction must be justified by an accepted definition, fact, or principle, such as the commutative or associative property of addition. For more about these properties of addition and their relationship to this theorem, see **Teacher Note,** Does the Order Matter? in *Stickers, Number Strings, and Story Problems.*

The model for such a notion of proof was first established by Euclid. In about 300 B.C., he codified what was known of ancient Greek geometry in his book, *Elements.* Euclid begins with the basic terms of geometry and, through hundreds of propositions and proofs, moves to beautiful and surprising theorems about geometric figures. What is remarkable is that, in each mathematical realm, you can get so far with such simple building blocks.

What Does Proof Look Like in Second Grade?

One does not expect the rigor of a formal proof, or the use of algebraic symbols and notation, from young students. Even for a mathematician, precise validation is often developed *after* new ideas have been explored thoroughly. When mathematical ideas are evolving, informal proofs are appropriate to communicate *why* a claim is true. These can include visual displays, concrete materials, or words. The test of the justification is this: Does it rely on mathematical logic rather than on one or a few examples to work?

In the Grade 2 discussion above, an important part of proving their idea is Henry's statement that it does not matter what number you start with. He understands that the procedure Alberto demonstrated with cubes can be used to show that the total number remains constant when 1 is taken from one addend and added to the other, no matter what two addends you choose.

When students make such claims of generality—*this is true for all numbers*—"all numbers" refers to the numbers they are using. In this vignette, the students' argument about addition concerns whole numbers. Although their reasoning applies equally well if working with integers or rational numbers, they will need to revisit this argument, using a different representation of number, when their domain of numbers expands beyond whole numbers.

When proving a generalization, students create models—for example, with cubes or diagrams—and use them to represent the operation. Then they reason about those models; for example, joining two sets of cubes to represent addition. In K–5 classrooms, in which mathematical ideas are generally "under construction," and in which sense-making and diverse modes of reasoning are valued, these are appropriate methods for making mathematical arguments.

The second graders' argument justifies the claim that if you subtract 1 from one addend and add 1 to another addend, the total remains the same. The sum of the numbers n and m is represented by the total number of cubes in two stacks. By moving one cube from one stack to another, Alberto demonstrated subtracting 1 from n and adding 1 to m. The total number of cubes remains unchanged; therefore, $n + m = (n - 1) + (m + 1)$. Alberto's demonstration, along with his classmates' explanations, validates the claim not only for particular numbers but for any whole numbers, and it easily conveys why the claim is true.

Second graders may extend this generalization to say that if *any amount* is subtracted from one addend and added to the other, the total stays the same (written algebraically, $n + m = (n - x) + (m + x)$). Some students use this generalization to solve addition problems, thinking, for example, $38 + 25 = 40 + 23 = 63$. They may prove that this works as Alberto did, but instead of moving one cube, they can move any number of cubes to prove that the total is unchanged.

To support this kind of reasoning, teachers should encourage students to use representations (cubes and number lines are two good options) to explain their thinking. This offers a reference for the student doing the explaining, and it also allows classmates to follow that reasoning. If students seem to be thinking only in terms of specific numbers, you may ask these questions: Will that work for other numbers? How do you know? Will the explanation be the same?