Teacher Note

Reasoning and Proof in Mathematics

As students find strategies to perform calculations, they frequently make claims about numerical relationships. Part of the work of fifth grade involves helping students to strengthen their ability to verbalize those claims and consider such questions as: Does this claim hold for all numbers? How can we know? Finding ways to answer these questions will provide the basis for making sense of formal proof when it is introduced in later years. Consider the following vignette in which students in a fifth grade class are developing methods for solving a subtraction problem.

Georgia: Here is what I tried for 1,232 − 196, I did what I do in addition. I changed the 196 to 200. 200 is easy to work with. 1,232 − 200 is 1,032 but now I am wondering if I should add the 4 and get 1,036 for the answer or subtract it to get 1,028.

Teacher: How would it work if you were adding 1,232 and 196?

Georgia: In addition I know I add 4 to one number and subtract 4 from the other and the answer is the same.

Teacher: Are you saying that if this were 1,232 + 196, you would change it to 1,228 + 200 by adding and subtracting 4 and know the answer would be the same?

Charles: You will get the same answer. If you take some number from one and put it on the other, the answer has to stay the same.

Rachel: I remember we used a story about apples to talk about this. It would be like I have 1,232 apples in one bag and 196 apples in another bag. I can take 4 of the apples out of the first bag and put them in the second bag. That means 1,232 + 196 = 1,228 + 200. You still have the same number of apples.

Teacher: Now Georgia is asking what happens if it isn't about addition. What if it is a subtraction problem? If you try to solve 1,232 − 196 by starting with 1,232 − 200, what do you do with the 4? Do you add it or subtract it to get the correct answer?

Georgia: Now I am thinking I should add the 4 to the 1,232 and make the problem 1,236 − 200. That will be the same as 1,232 − 196. Whatever you add to one you have to add to the other. I see it now. It keeps them the same amount apart.

Rachel: I think that will be true all the time for subtraction—you add the same to both.

Teacher: How could you show whether what Rachel says is always true? Let's first be clear about what Rachel is saying.

The teacher asks Rachel to repeat her assertion and writes Rachel's words on the board:

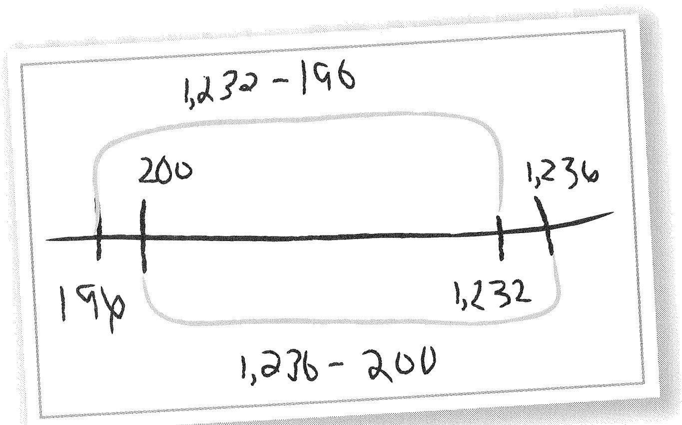
> In a subtraction problem, you can add the same amount to both numbers and the answer will still be the same.

Teacher: I'd like you to show whether or not what Rachel is saying is true. You can start by trying some more examples, but then I want you to use a story problem, like Rachel's apple story, or a number line, or some kind of picture to show your thinking. You can use story problems, number lines, or diagrams to show your thinking. Remember, Rachel says this works for all subtraction problems. You have to think about how your explanation works for all numbers.

In this class, Rachel has made an assertion—mathematicians call such an assertion a conjecture—that if you add a certain amount to both numbers in a subtraction problem, the difference remains the same. The teacher has challenged the class to find a way to show that this conjecture is true.

Let us return to the Grade 5 class to see how the students responded to the teacher's challenge to justify their conjecture.

Martin: I made a number line. On the top is 1,232 − 196. The bottom shows 1,236 − 200. You just move the whole thing over 4. It has to be the same.



Martin's Work

Kim: I thought of a story. My little sister and I both have penny banks. I have 1,232 pennies and she has 196. To see how many more pennies I have, I'd do 1,232 − 196. But suppose my mom gives us each 4 pennies. Then it would be 1,236 − 200. I still have the same amount more.

Teacher: Kim's story and Martin's number line both show us that 1,232 − 196 = 1,236 − 200. How can we use their work to say this works for all numbers?

Felix: With the number line it doesn't matter what the numbers are at the beginning or the end of the jump. If you move the loop to the right or to the left, the loop stays the same.

Cecelia: It is the same with Kim's story. Whatever is in the penny banks at the beginning, you can add the same number to both of them and the difference would have to be the same.

Kim has used a story context that represents subtraction, in this case a comparison situation, to explain how adding the same amount to both numbers in a subtraction problem will keep the difference the same. Martin has drawn a number line to illustrate the same relationship. Felix then explains how Martin's number line can illustrate any subtraction problem. Similarly Cecelia explains how Kim's story context can apply to all subtraction problems.

Students in Grades K–5 can work productively on developing justifications for mathematical ideas as the students in this class do here. But what is necessary to prove an idea in mathematics? First we'll examine what proving is in the field of mathematics, then we will return to the kind of proving students can do in fifth grade.

What is Proof in Mathematics?

Throughout life, when we make a claim or assertion, we are often required to justify the claim to persuade others that it is valid. A prosecutor who claims that a defendant is guilty of a crime, must make an argument, based on evidence, to convince the jury of this claim. A scientist who asserts that the earth's atmosphere is becoming warmer must marshal evidence, usually in the form of data and accepted theories and models, to justify the claim. Every field, including the law, science, and mathematics, has its own accepted standards and rules for how a claim must be justified in order to persuade others.

In mathematics, a *theorem* must start with a mathematical assertion, which has explicit hypotheses (or "givens") and an explicit conclusion. The proof of the theorem must show how the conclusion follows logically from the hypotheses. A mathematical argument is based on logic and gives a sense of why a proposition is true. For instance, Georgia and Rachel asserted that the difference of two numbers remains the same if you add the same amount to both. In later years, Georgia's statement might be expressed as $(a - b) = (a + n) - (b + n)$. The proof of this claim consists of a series of steps in which one begins with the hypothesis—

a and b are numbers—and follows a chain of logical deductions ending with the conclusion—$(a - b) = (a + n) - (b + n)$. Each deduction must be justified by an accepted definition, fact, or principle, such as the commutative and associative laws of addition and the laws describing additive inverses and identities.

The model for such a notion of proof was first established by Euclid, who codified what was known of Ancient Greek geometry in his *Elements,* written about 300 B.C. In his book, Euclid begins with the basic terms of geometry (a point, a line) and their properties (a line is determined by two points) and, through hundreds of propositions and proofs, moves to beautiful and surprising theorems about geometric figures.

What Does Proof Look Like in Fifth Grade?

One does not expect the rigor or sophistication of a formal proof or the use of algebraic symbolism from children in the elementary grades. Even for a mathematician, precise validation is often developed *after* new mathematical ideas have been explored and are more solidly understood. When mathematical ideas are evolving and there is a need to communicate the sense of *why* a claim is true, then informal methods of proving are appropriate. Such methods can include the use of visual displays, concrete materials, or words. The test of effectiveness of such a justification is: Does it rely on logical thinking about the mathematical relationships rather than on the fact that one or a few specific examples work?

An important part of the fifth grader's justification is Felix's statement that it doesn't matter what the numbers are. He understands how Martin's representation of subtraction on the number line could apply to any subtraction situation. In a similar way, Cecelia is able to use the representation of subtraction in Kim's comparison story to reason about other subtraction problems.

Proving, by calling upon a model that represents the operation as these students do, is particularly appropriate in K–5 classrooms where mathematical ideas are generally "under construction," and in which sense-making and diverse modes of reasoning are valued. The fifth graders' argument offers justification for the claim that if you add the same amount to both numbers in a subtraction problem, the difference remains the same. For Kim, the difference is represented by comparing the two amounts in penny banks. Adding the same amount to both banks will not change the difference. Kim's argument not only establishes the validity of the claim for particular numbers, but for any whole numbers, and easily conveys why it is true. Martin's number line diagram offers a visual image for the subtraction situation; the jump represents the difference between any two given numbers.

To support the kind of reasoning illustrated in the vignette, teachers should encourage students to use representations such as cubes, story contexts, or number lines to explain their thinking. The use of representations offers a reference for the student who is explaining his or her reasoning, and it also allows more classmates to follow that reasoning. If it seems that students may be thinking only in terms of specific numbers, teachers might ask,

Will that work for other numbers? How do you know? Will the explanation be the same?